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Quantum Cosmology: A Brief Review and Investigation of Coupling Spinors to Gravity

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Abstract

This dissertation tells a story about gauge theories and spinors from the early 1920s to the late 1980s. The first chapter investigates the meaning behind gauge theories and gives a concrete picture of what is meant by gauge symmetry. We later investigate the inception of gauge invariance in the context of Weyl's unified approach to gravity. A later section introduces torsion and the significance of boundary terms. Specifically, in Einstien's teleparallel theory, an overlooked boundary term in the original formalism provides a new look at extrinsic curvature in the presence of torsion. We investigate the geometric aspect of gauge theories in the formal language of fibre bundles and apply this notion to gravity. The original ECKS approach of coupling spinors to gravity is modernised using the first-order gravity formulation found in supergravity. We also highlight problems with translations of the Poincare group in ECKS. The final chapter reviews subsequent work by Professor Kellogg Stelle and Professor Arkady Tseytlin in symmetry breaking from the (Anti-)De Sitter to Lorentz group, inducing gravity. It shows how some issues with the Poincare transformations found in ECKS are explained in a rigorous non-linear framework.

Acknowledgement

Many thanks to my supervisor, Professor João Magueijo for the support and insightful meetings over the past months. Also a special thanks to Professor Travis Schedler for a very useful maths email correspondence. Additionally, I extend my thanks to the rest of the QFFF faculty for an enriching year, I can honestly say I've learnt more in this year than I ever have. Finally a special thanks to my friends and family for helping and supporting me throughout this academic year.

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Chapter 1

Introduction

1.1 Emergence of Gauge theories

1.1.1 Demystifying Gauge Symmetries

Gauge theories generally have a somewhat paradoxical importance in physics; many say gauge symmetries do not change the underlying physics, but in the same breath describe them as the bedrock of current physics. To understand gauge theories, local and global transformations, redundancies, and symmetries must be well defined. For this discussion, symmetry breaking is left out as this is beyond the scope of this dissertation. Much of this discussion is a review of [34].

To give an intuitive description, imagine you have a perfectly spherical and uniform ball in front of you. If you close your eyes and someone performs a rotation transformation on the ball, when you observe the ball again, you won't be able to tell the difference before or after rotation. This means rotations are a symmetry of the ball.

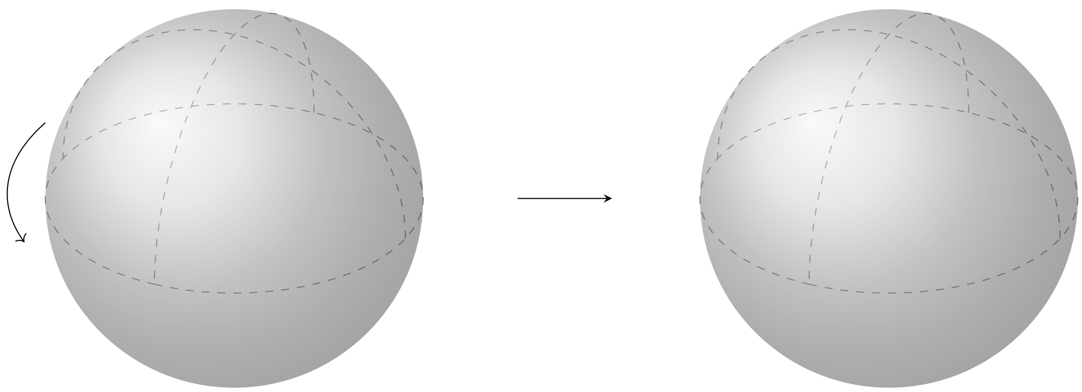

Figure 1.1: Rotation of a uniform sphere remains identical before and after rotation

Put succinctly, a symmetry changes the system from one state to another without changing the properties of the system. Using this definition, symmetries are observable attributes of a system. On the other hand, a **redundancy** changes the description of a system without changing the state of said system.

Let's say your system is now a box with a falling ball and a stall. In this context, a **global transformation** would be rotations of the entire system by 45 degrees without changing the physics on the inside.

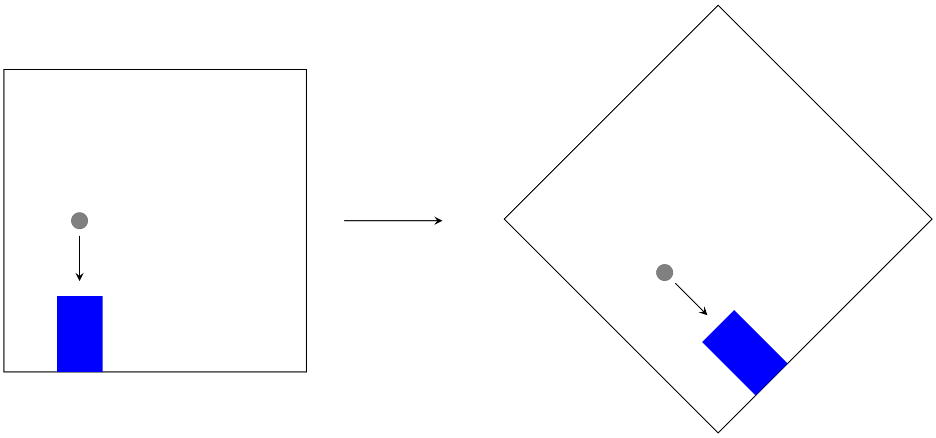

Figure 1.2: Rotating a ball falling in a box by 45 degrees, demonstrating a global transformation

As you can see from the perspective of someone inside the box, there is no perceived change in the physics in either state; hence, the system is deemed to have a global rotational symmetry. You could argue that the systems are not identical from an outside observer's perspective; however, this is irrelevant for physicists as we live inside the "box" (universe).

Now, instead of rotating the entire system, rotate the blue stall. This is an example of a local transformation.

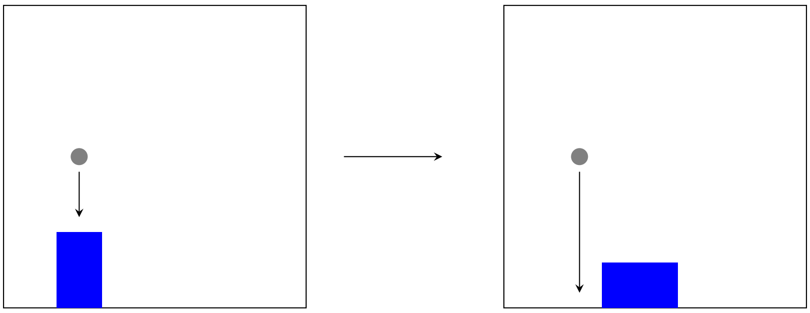

Figure 1.3: Rotating a stall in a box by 45 degrees, demonstrating a local transformation

As you can see, if you rotate the block on the inside, the result will be the ball smacking the floor instead of the stall; hence, from the perspective of an observer inside the box, physics has changed.

Before continuing, a distinction between transformations is required. Thus far, only active transformations have been considered. Instead, let's look at a different type of transformation. One which merely changes the way we describe a system. Take, for example, a ball on a number line, where we consider the number line to be our coordinate system. Moving the ball is an active transformation, as the ball has moved. On the other hand, moving the number line is a passive transformation, as this changes the description/coordinate of the ball but doesn't move it.

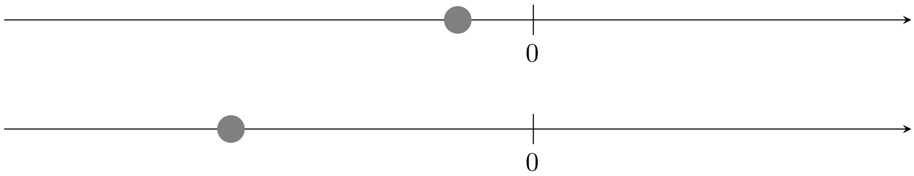

Figure 1.4: Active transformation by moving the ball

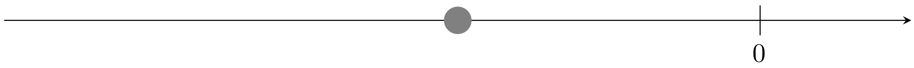

Figure 1.5: [34] Passive transformation by moving the number line

A system that is invariant under a passive transformation has **redundancies** and under an active transformation **symmetries**.

By extending mathematical formalisms, redundancies are introduced in the systems descriptions. To track these redundancies the notion of a "bookkeeper" [34] is introduced. The distinguishing feature of an active and passive transformation is that the bookkeeper changes automatically with a passive transformation, however this isn't always the case for active transformations.

1.1.2 Weyl's Unified Theory

Before getting into the nitty-gritty EC formalism and coupling spinors to gravity, one must address the vast body of work that came before. The aim of the physicists at the time was to unify General Relativity and electromagnetism. Starting in 1918, the German physicist, Herman Weyl, wondered if the magnitude of a vector in Riemann space should remain constant along a given path (parallel transport).

Definition (Parallel Transport [3]): Let $x^\alpha(\lambda)$ be an affinely parameterised worldline with the tangent vector $\frac{dx^\alpha}{d\lambda}$. A vector, S^β, is parallelly transported along the worldline if:

$$\frac{dx^\alpha}{d\lambda}\nabla_\alpha S^\beta = 0 \tag{1.1}$$

Loosening the metricity condition, he introduced a one-form to the Levi-Civita connection. Weyl postulated this was the Coulomb potential [36]:

$$\nabla_\alpha g_{\mu\nu} = A_\alpha g_{\mu\nu} \tag{1.2}$$

Theorem 1 (Uniqueness of the covariant derivative [3])**.** *Let ∇_μ be the covariant derivative on some manifold $\mathcal{M}$. If the covariant derivative is torsion-free, then the connection coefficients are uniquely given by the Christoffel symbol and non-metricity condition.*

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\rho_{\alpha\mu} g_{\nu\rho} - \Gamma^\rho_{\alpha\nu} g_{\mu\rho} = A_\alpha g_{\mu\nu} \tag{1.3}$$

$$\nabla_\nu g_{\mu\alpha} = \partial_\nu g_{\mu\alpha} - \Gamma^\rho_{\nu\mu} g_{\alpha\rho} - \Gamma^\rho_{\nu\alpha} g_{\mu\rho} = A_\nu g_{\mu\alpha} \tag{1.4}$$

$$\nabla_\mu g_{\alpha\nu} = \partial_\mu g_{\alpha\nu} - \Gamma^\rho_{\mu\alpha} g_{\nu\rho} - \Gamma^\rho_{\mu\nu} g_{\alpha\rho} = A_\mu g_{\alpha\nu} \tag{1.5}$$

Taking (1.2) + (1.3) - (1.4) reveals the connection is [32]:

$$\Gamma^\alpha_{\beta\mu} = \left\{ {\alpha \atop \beta\mu} \right\} - \frac{1}{2} g^{\alpha\rho} \left(A_\beta g_{\rho\mu} + A_\mu g_{\rho\beta} - A_\rho g_{\mu\beta} \right) \tag{1.6}$$

Where $\left\{ {\alpha \atop \beta\mu} \right\}$ is the connection for Riemannian geometry. Performing what he named a "gauge transformation". Weyl re-scaled the metric tensor and added a term to the Coulomb potential leaving the connection invariant [32]. Therefore, the geodesic equations are invariant under Weyl transformations [36]:

$$g^{\mu\nu} \to \Lambda(x)\, g^{\mu\nu} \tag{1.7}$$

$$\Lambda(x) = \exp \lambda(x) \tag{1.8}$$

$$A_\alpha \to A_\alpha + \partial_\alpha \lambda \tag{1.9}$$

$$\Gamma^\alpha_{\beta\mu} \to \Gamma^\alpha_{\beta\mu} \tag{1.10}$$

As long as physical observables are left invariant, gauge transformations are permissible. This encapsulates the goal of gauge theory, finding the dynamics of redundant (nonphysical) degrees of freedom. This theory, unfortunately, fell flat on its head. Einstein pointed out: the proper time interval varies under a Weyl transformation.

Definition (Proper time interval [3]): The proper time interval, $\Delta\tau$, is the time interval measured by a clock travelling along a parallel worldline. In mathematical terms:

$$\Delta\tau_{ab} = \int_{\lambda_a}^{\lambda_b} \left(g_{\mu\nu} T^\mu T^\nu \right)^{\frac{1}{2}} d\lambda \tag{1.11}$$

Where $T^\mu = \frac{dx^\mu}{d\lambda}$, is a tangent vector to the worldline $x^\mu(\lambda)$.

To illustrate this using (1.1), the infinitesimal change in a vector along a worldline is:

$$\frac{dS^{\beta}}{d\lambda} = -\Gamma^{\beta}_{\rho\alpha} S^{\rho} \frac{dx^{\alpha}}{d\lambda} \tag{1.12}$$

Plugging (1.11) into the definition of vector magnitude [32]:

$$\frac{d}{d\lambda}\left(L^2\right) = \frac{d}{d\lambda}\left(g_{\mu\nu} L^{\mu} L^{\nu}\right) \tag{1.13}$$

$$\frac{dL}{d\lambda} = L \frac{A_{\alpha}}{2} \frac{dx^{\alpha}}{d\lambda} \tag{1.14}$$

Integrating from some initial position $x_0(\lambda_0)$ to $x_1(\lambda_1)$ on the worldline x^{α} implies [32]:

$$L_1 = L_0 \exp \int_{\vec{x_0}}^{\vec{x_1}} \frac{A_{\alpha}}{2} dx^{\alpha} \tag{1.15}$$

This would mean the proper time interval, $\Delta\tau$, depends on the non-metricity condition.

$$\Delta\tau_{01} = \int_{\lambda_0}^{\lambda_1} T_0 \left(\exp \int_{\vec{x_0}(\lambda_0)}^{\vec{x_1}(\lambda)} \frac{A_{\alpha}}{2} dx^{\alpha} \right)^{\frac{1}{2}} d\lambda \tag{1.16}$$

$$T_0 = \left(g_{\mu\nu} T^{\mu} T^{\nu}|_{\lambda_0}\right)^{\frac{1}{2}} \tag{1.17}$$

Hence violating the principle of *Gauge Invariance*.

Side Note: There are propositions to modify the proper time to remove the exponential, however, this is beside the point of gauge theories. If a set of transformations modify physical quantities, then gauge invariance is broken and one should move on. Modifying the metricity condition is the cause of this theory's downfall and in my opinion should be avoided.

Despite the theories' downfall, the inception of gauge transformations led to scientific breakthroughs in particle physics. Later in 1929, Weyl revisited gauge transformation in Quantum mechanics. He believed phase rotational invariance manifested as a local phenomenon in quantum field theory. Weyl looked at the Dirac Lagrangian, $\mathcal{L}$, and considered a global phase transformation of the electron's wavefunction.

$$\mathcal{L} = \bar{\Psi}(\gamma^{\mu}\partial_{\mu} - m)\Psi \tag{1.18}$$

$$\Psi \to e^{i\lambda}\Psi,\ \bar{\Psi} \to e^{-i\lambda}\Psi \tag{1.19}$$

$$\mathcal{L} \to \mathcal{L} \tag{1.20}$$

Where λ is a constant here. This transformation is called a global $U(1)$ transformation and hence $U(1)$ is a *symmetry* of the Dirac Lagrangian. Analogues to 1.2, consider an electron beam hitting an isolated diffraction grating apparatus. A global phase transformation, physically speaking, would be placing a phase shifter before the electron beam hits the diffraction grating. The diagram below depicts what is occurring:

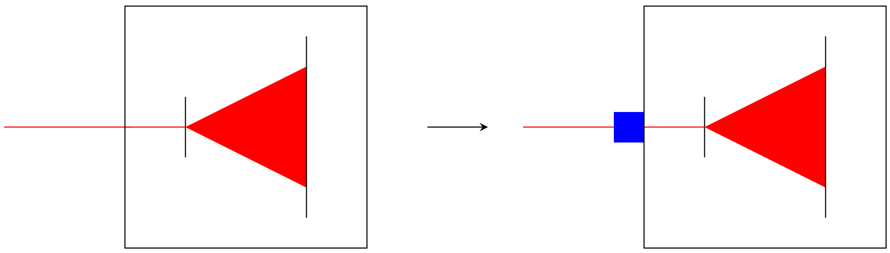

Figure 1.6: An active global phase transformation of an electron beam (red) before diffraction (first black line). The image on the left is the diffraction grating experiment with no phase transformation. The diagram on the right is after a global phase change (blue box). The red triangle represents the diffracted electron beam.

Since the equations of motion for the electron remain the same under a global phase transformation, an observer inside the box will not see any change in the diffraction grating experiment. Now consider a local phase transformation, where $\lambda \to \lambda(x)$

$$\Psi \to e^{i\lambda(x)}\Psi \tag{1.21}$$

$$\partial_{\mu}\Psi \to e^{i\lambda}(\partial_{\mu}\Psi + i\Psi\partial_{\mu}\lambda) \tag{1.22}$$

$$\mathcal{L} \not\to \mathcal{L} \tag{1.23}$$

Because of the extra x dependence in λ the Dirac Lagrangian is not invariant under an active local $U(1)$ transformation. This local phase transformation presents itself in the physical world by placing a phase shift after the diffraction grating. After the diffraction grating the electron's wavefunction disperses so the phase transition only occurs on part of the wavefunction, hence $\lambda(x)$ pics up a positional dependence.

The local transformation looks as follows:

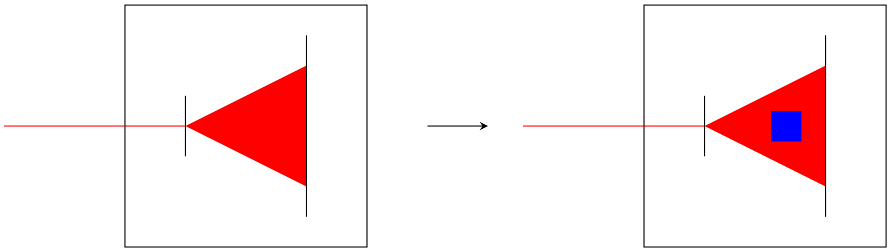

Figure 1.7: An active local phase transformation of an electron beam (red) before diffraction (first black line). The image on the left is the diffraction grating experiment with no phase transformation. The diagram on the right is after a local phase change (blue box). The red triangle represents the diffracted electron beam.

A vector field, A, and covariant derivative, D, are introduced to remove invariant terms. The field, A acts as a dynamic bookkeeper of local transformation, allowing physicists to view the local transformation as merely a passive transformation. This reveals a $U(1)$ local *redundancy* in the Dirac Lagrangian.

$$A_\mu \to A_\mu - \partial_\mu \lambda \tag{1.24}$$

$$D_\mu = \partial_\mu + iA_\mu \tag{1.25}$$

$$\mathcal{L} \to \mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \tag{1.26}$$

These transformations removed the physical nature of a local transformation, so instead of viewing the local transformation as active (presented in 1.7), the local transformation is merely a change in local coordinates. So, in physics, this is a redundancy in the description when referring to a *local gauge symmetry*. Much like the Levi-Civita connection defined in GR, the bookkeeper A is also a connection with an associated curvature F. In the Yang-Mills section, this is defined in great detail using the language of differential geometry and fibre bundles. Taking this a step further and including a kinetic term in the Dirac Lagrangian yields the QED Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \tag{1.27}$$

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} \tag{1.28}$$

Theorem 2 (Noether's Theorem)**.** *Every Lie Group symmetry of the Lagrangian gives a corresponding conservation law.*

By varying the QED action w.r.t A you find:

$$\boldsymbol{\delta S}[\boldsymbol{A}, \Psi; \boldsymbol{\delta A}] = \int d^4x(\partial_\nu F^{\nu\mu} + \underbrace{\bar{\Psi} i\gamma^\mu \Psi}_{-J^\mu})\delta A_\mu \tag{1.29}$$

$$\partial_\nu F^{\nu\mu} = J^\mu \tag{1.30}$$

$$\partial_\mu \partial_\nu F^{\nu\mu} = 0 \tag{1.31}$$

This shows the true power of gauge theory and Noether's theorem; simply through symmetry, you can find all the local dynamics of a theory. Noether currents act as a rule book for gauge particles to follow.

1.2 Einstien-Cartan and Teleparralelism

Einstein's approach followed Weyl's trail of thought, instead of loosening the metricity condition, torsion was the next candidate [43]. These theories, including Weyl's, are an extension of Riemannian geometry. The aim was to extend the number of degrees of freedom to include electromagnetism. From 1925-1930 Einstien developed his theory of Teleparallel; publishing the following papers on the matter [6] [8] [9] [7].

Einstein began by considering definition (1.1) in the context of a "4-dimensional continuum"[43] and affine connection.

Definition (Continuum/C^∞ n-Manifold [2][41]): Let $\mathcal{M} = \bigcup_{i\in I} U_i$ such that $U_i \subset \mathcal{M}$ and $dim(\mathcal{M}) = n$. Let $\phi : U_i \to \mathbb{R}^n$ be a *bijective* function such that $\phi_i(U_i)$ is open. $\forall i, j \in I$, one has $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ has continuous partial derivatives. $V \subset \mathcal{M}$ is open if $\phi_i(U_i \cap V)$ is open for all $i \in I$. Assuming the topology of $\mathcal{M}$ is Hausdorff, $\{\phi_i | i \in I\}$ is called an atlas of $\mathcal{M}$. The equivalence class of atlases forms a differentiable manifold. Any $\phi_i : U_i \to \mathbb{R}^n$ is called a chart/coordinate system.

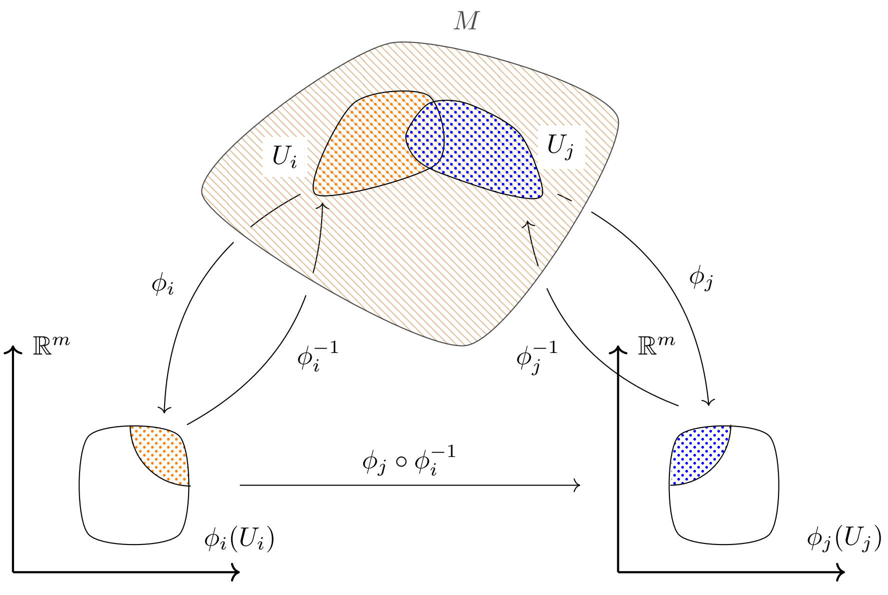

Figure 1.8: Diagram of open coverings mapping from a manifold to $\mathbb{R}^n$

Consider the vector, S^μ, infinitesimally pushed along a geodesic [3]:

$$(\boldsymbol{d}S)^\mu = -\Gamma^\mu_{\alpha\nu} S^\alpha \boldsymbol{d}x^\nu \tag{1.32}$$

Einstein asked what if the connection isn't fully symmetric about α and ν? Leaving the symmetry of the connection ambiguous is the punchline of his research. Including an anti-symmetric component reveals a new field called torsion.

$$T_{\rho\mu}{}^{\beta} = 2\Gamma^{\beta}_{[\rho\mu]} \tag{1.33}$$

The logical question to ask at this point is why is the difference between two connections, which are not tensors, equal to a tensor?

$$\Gamma'^{\sigma}_{\alpha\mu} = \frac{\partial x'^{\sigma}}{\partial x^{\beta}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\tau}}{\partial x'^{\alpha}}\Gamma^{\beta}_{\tau\rho} + \frac{\partial x'^{\sigma}}{\partial x^{\beta}}\frac{\partial^2 x^{\beta}}{\partial x'^{\alpha}\partial x'^{\mu}} \tag{1.34}$$

$$\Gamma'^{\sigma}_{\alpha\mu} - \Gamma'^{\sigma}_{\mu\alpha} = \frac{\partial x'^{\sigma}}{\partial x^{\beta}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\tau}}{\partial x'^{\alpha}}\left(\Gamma^{\beta}_{\tau\rho} - \Gamma^{\beta}_{\rho\tau}\right) \tag{1.35}$$

Hence there is no contradiction. Following Theorem 1, if the metricity condition holds the connection coefficients are uniquely given by the Christoffel symbol and torsion tensor:

$$\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu} - \Gamma^{\rho}_{\alpha\mu} g_{\rho\nu} - \Gamma^{\rho}_{\alpha\nu} g_{\rho\mu} - \Gamma^{\rho}_{\nu\alpha} g_{\rho\mu} \tag{1.36}$$

$$-\Gamma^{\rho}_{\nu\mu} g_{\rho\alpha} + \Gamma^{\rho}_{\mu\nu} g_{\rho\alpha} + \Gamma^{\rho}_{\mu\alpha} g_{\rho\nu} = 0$$

$$\Gamma^{\beta}_{\alpha\nu} = \left\{{\beta \atop \alpha\nu}\right\} + \frac{1}{2}\left(T^{\beta}{}_{\alpha\nu} - T_{\nu}{}^{\beta}{}_{\alpha} + T_{\alpha\nu}{}^{\beta}\right) \tag{1.37}$$

$$= \left\{{\beta \atop \alpha\nu}\right\} + K_{\alpha\nu}{}^{\beta} \tag{1.38}$$

Where $K_{\alpha\nu}{}^{\beta}$ is the contorsion tensor.

One might naively think the connection's symmetric part is left untouched; hence, the geodesic equations remain the same. However, this is far from the truth. The symmetric part of the connection is [3]:

$$\Gamma^{\beta}_{(\alpha\nu)} = \left\{{\beta \atop \alpha\nu}\right\} + T_{(\alpha\nu)}{}^{\beta} \tag{1.39}$$

1.2.1 Torsion and curvature

Geometrically speaking, what does a connection with torsion mean?

The following derivation was adapted from [26] and [41]. Defining an infinitesimal parallelogram $ABCD$ on a differentiable manifold M. The coordinates of each point are $\{x^\mu(A)\}$, $\{x^\mu(A) + \epsilon^\mu_A\}$, $\{x^\mu(A) + \epsilon^\mu_A + \delta^\mu_A\}$ and $\{x^\mu(A) + \delta^\mu_A\}$ respectively.

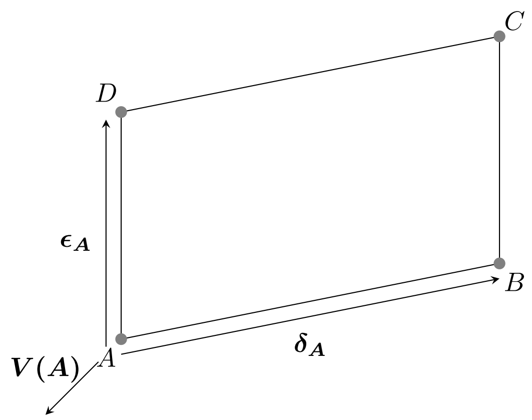

Figure 1.9: Infinitesimal parallelogram for a connection endowed with torsion.

Defining the first path as ABC, if the vector $\boldsymbol{V}(A) \in T_A M$ is parallelly transported to B along ABC then $\boldsymbol{V_{ABC}}(B) \in T_B M$ is denoted as:

$$V^{\mu}_{ABC}(B) = V^{\mu}(A) \underbrace{-V^{\beta}(A)\Gamma^{\mu}_{\nu\beta}(A)\epsilon^{\nu}}_{dV^{\beta}(A)} \tag{1.40}$$

Then parallelly transporting to C:

$$\begin{aligned} V^{\mu}_{ABC}(C) &= V^{\mu}_{ABC}(B) - V^{\beta}_{ABC}(B)\Gamma^{\mu}_{\nu\beta}(B)\delta^{\nu} && (1.41)\\ &= V^{\mu}(A) - V^{\beta}(A)\Gamma^{\mu}_{\nu\beta}(A)\epsilon^{\nu} - \left(V^{\beta}(A) - V^{\rho}(A)\Gamma^{\beta}_{\nu\rho}(A)\epsilon^{\nu}\right) && (1.42)\\ &\quad \times \left(\Gamma^{\mu}_{\lambda\beta}(A) + \partial_{\alpha}\Gamma^{\mu}_{\lambda\beta}(A)\epsilon^{\alpha}\right)\delta^{\lambda} \\ &= V^{\mu}(A) - V^{\beta}(A)\Gamma^{\mu}_{\nu\beta}(A)\epsilon^{\nu} - V^{\beta}(A)\Gamma^{\mu}_{\lambda\beta}(A)\delta^{\lambda} && (1.43)\\ &\quad - V^{\beta}(A)\left(\partial_{\alpha}\Gamma^{\mu}_{\lambda\beta}(A) - \Gamma^{\rho}_{\alpha\beta}(A)\Gamma^{\mu}_{\lambda\rho}(A)\right)\epsilon^{\alpha}\delta^{\lambda} + \mathcal{O}\left(\epsilon^2, \delta^2\right) \end{aligned}$$

Where in 1.42, $\Gamma(B)$ was taylor expanded about ϵ. Performing the same calculation for the path ADC one finds:

$$\begin{aligned} V^{\mu}_{ADC}(C) = V^{\mu}(A) - V^{\beta}(A)\Gamma^{\mu}_{\nu\beta}(A)\epsilon^{\nu} - V^{\beta}(A)\Gamma^{\mu}_{\lambda\beta}(A)\delta^{\lambda} \\ -V^{\beta}(A)\left(\partial_{\alpha}\Gamma^{\mu}_{\lambda\beta}(A) - \Gamma^{\rho}_{\alpha\beta}(A)\Gamma^{\mu}_{\lambda\rho}(A)\right)\delta^{\alpha}\epsilon^{\lambda} + \mathcal{O}\left(\epsilon^2, \delta^2\right) \end{aligned} \tag{1.44}$$

Taking the difference between the two vectors at C:

$$V^{\mu}_{ADC}(C) - V^{\mu}_{ABC}(C) = V^{\beta}(A)\Big(-\partial_{\alpha}\Gamma^{\mu}_{\lambda\beta}(A) + \Gamma^{\rho}_{\alpha\beta}(A)\Gamma^{\mu}_{\lambda\rho}(A) \tag{1.45}$$

$$+ \partial_{\alpha}\Gamma^{\mu}_{\lambda\beta}(A) - \Gamma^{\rho}_{\alpha\beta}(A)\Gamma^{\mu}_{\lambda\rho}(A)\Big)\delta^{\alpha}\epsilon^{\lambda}$$

$$= V^{\beta}(A)R^{\mu}{}_{\beta\lambda\alpha}\delta^{\alpha}\epsilon^{\lambda} \tag{1.46}$$

Hence the Riemann curvature tensor is defined in the usual sense:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{1.47}$$

Looking at the structure of 1.28 there are notable similarities between the Field strength tensor and the Riemann tensor. As we'll see later, this leads to a compelling and natural argument for coupling spinors to gravity.

The way curvature manifests itself is by "rolling" the tangent space over the manifold, on the other hand, torsion introduces a "twist" [11] to this tangent space. Adapting calculations from David Tong's lecture series on General relativity [41]: Take two infinitesimal vectors $\boldsymbol{\epsilon_p} = \epsilon^{\mu}\boldsymbol{\partial_{\mu}}$, $\boldsymbol{\delta_p} = \delta^{\mu}\boldsymbol{\partial_{\mu}} \in T_pM$ and parallel transport the vectors along each other s.t:

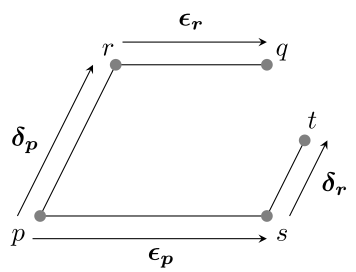

Figure 1.10: [41] Image of infinitesimal open parallelogram due to torsion.

$$\epsilon^{\mu}_{r} = \epsilon^{\mu}_{p} - \Gamma^{\mu}_{\nu\rho}(p)\delta^{\nu}_{p}\epsilon^{\rho}_{p} \tag{1.48}$$

Where r has coordinates $x^{\mu}(p) + \delta^{\mu}_{p}$. Similarly, the point calculating the parallel transport of δ_p along ϵ_p:

$$\delta^{\mu}_{s} = \delta^{\mu}_{p} - \Gamma^{\mu}_{\nu\rho}(p)\epsilon^{\nu}_{p}\delta^{\rho}_{p} \tag{1.49}$$

Where s has coordinates $x^{\mu}(p) + \epsilon^{\mu}_{p}$. Now if you parallel transport any vector along $\boldsymbol{\delta_s}$ and $\boldsymbol{\epsilon_r}$ you end up with the points respectively q and t with coordinates:

$$q : x^{\mu}(q) = x^{\mu}(p) + \delta^{\mu}_{p} + \epsilon^{\mu}_{r} \tag{1.50}$$
$$= x^{\mu}(p) + \delta^{\mu}_{p}\epsilon^{\mu}_{p} - \Gamma^{\mu}_{\nu\rho}(p)\delta^{\nu}_{p}\epsilon^{\rho}_{p} \tag{1.51}$$
$$t : x^{\mu}(t) = x^{\mu}(p) + \epsilon^{\mu}_{p} + \delta^{\mu}_{s} \tag{1.52}$$
$$= x^{\mu}(p) + \epsilon^{\mu}_{p} + \delta^{\mu}_{p} - \Gamma^{\mu}_{\nu\rho}(p)\epsilon^{\nu}_{p}\delta^{\rho}_{p} \tag{1.53}$$

As you can now see the difference in coordinates $x^{\mu}(q)$ and $x^{\mu}(t)$ is torsion tensor:

$$x^{\mu}(q) - x^{\mu}(t) = T_{\nu\rho}{}^{\mu}\epsilon^{\nu}_{p}\delta^{\rho}_{p} \tag{1.54}$$

Visually speaking, "torsion measures the failure of the parallelogram to close" [41].

1.2.2 Einstien Hilbert action

Einstein's next constructed the Einstein-Hilbert action [16]. Since these are Einstein's equations with no matter content, there is a neat trick where you vary w.r.t the tensor density of the metric instead [6]:

$$\boldsymbol{S} = \int R_{\mu\nu}\boldsymbol{g}^{\mu\nu}d^4x \tag{1.55}$$
$$\boldsymbol{\delta S}\left[\boldsymbol{\delta g};\boldsymbol{\Gamma},\boldsymbol{g}\right] = \int \left(R_{\mu\nu}\right)\delta\boldsymbol{g}^{\mu\nu}d^4x \tag{1.56}$$
$$R_{\mu\nu} = 0 \tag{1.57}$$
$$\dot{R}_{\mu\nu} + \partial_{\rho}K_{\nu\mu}{}^{\rho} - \partial_{\nu}K_{\rho\mu}{}^{\rho} + K_{\rho\lambda}{}^{\rho}K_{\nu\mu}{}^{\lambda} - K_{\nu\lambda}{}^{\rho}K_{\rho\mu}{}^{\lambda} = 0 \tag{1.58}$$

Where the Ricci tensor made from the levi civita symbol is $\dot{R}_{\mu\nu}$. From the Palitani formalism, it is clear even the original Einstein equations are modified.

As every good physicist does, Einstein neglected a boundary term.

Inspired by the Gibbons–Hawking–York boundary term [48][12][14], the action is redefined similarly. This may seem pedantic; however, if you wanted to perform path integrals on $\mathcal{M}$, $\mathcal{M}$ must have no boundary. In neglecting this term Einstien has implicitly assumed a *closed* universe. As will be demonstrated later, consider a spacetime foliated by time. If each time leaf is *compact* (no boundary), then the Hamiltonian evaluates to zero. Therefore, the total energy inside the universe, even when considering matter, evaluates to zero.

$$\begin{aligned}\boldsymbol{\delta S}\left[\boldsymbol{\delta\Gamma};\boldsymbol{\Gamma},\boldsymbol{g}\right] &= \int_{\mathcal{M}} \left(\partial_\rho \delta\Gamma^\rho{}_{\nu\mu} - \partial_\nu \delta\Gamma^\rho{}_{\rho\mu} + \delta\Gamma^\rho{}_{\rho\lambda}\Gamma^\lambda_{\nu\mu}\right. && (1.59)\\ &\quad \left.+ \Gamma^\rho_{\rho\lambda}\delta\Gamma^\lambda{}_{\nu\mu} - \delta\Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda_{\rho\mu} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda{}_{\rho\mu}\right)\boldsymbol{g}^{\mu\nu} d^4x \\ &= \int_{\mathcal{M}} -\partial_\rho \boldsymbol{g}^{\mu\nu}\delta\Gamma^\rho{}_{\nu\mu} + \partial_\nu \boldsymbol{g}^{\mu\nu}\delta\Gamma^\rho{}_{\rho\mu} && (1.60)\\ &\quad + \left(\delta\Gamma^\rho{}_{\rho\lambda}\Gamma^\lambda_{\nu\mu} + \Gamma^\rho_{\rho\lambda}\delta\Gamma^\lambda{}_{\nu\mu} - \delta\Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda_{\rho\mu} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda{}_{\rho\mu}\right)\boldsymbol{g}^{\mu\nu} \\ &\quad + \partial_\rho\left(\delta\Gamma^\rho{}_{\nu\mu}\boldsymbol{g}^{\mu\nu} - \delta\Gamma^\nu{}_{\nu\mu}\boldsymbol{g}^{\mu\rho}\right) d^4x\end{aligned}$$

Side Note: $\delta\Gamma^\alpha{}_{\nu\mu}$ is a tensor as the variation of the Christofell symbol is a tensor, and the variation of torsion is a tensor, hence the difference in notation.

The full derivative term is called the Palatini identity. Remarkably, Einstein discovered this well before Palatini did in his first paper on teleparallelism. The power of the Palatini identity is explicitly seen here, as you can derive equations of motion on general geometries without explicitly defining the connection. At this point, one needs to be careful when deriving the boundary, $\partial\mathcal{M}$. Using Stokes's theorem on the full derivative term gives:

$$\boldsymbol{\delta S}\left[\boldsymbol{\delta\Gamma};\boldsymbol{\Gamma},\boldsymbol{g}\right] = \oint_{\partial\mathcal{M}} \mathrm{d}^3y\sqrt{|h|}n_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda{}_{\nu\mu} - g^{\mu\lambda}\delta\Gamma^\nu{}_{\nu\mu}\right)\big|_{\partial\mathcal{M}} + \dots \tag{1.61}$$

Where $h_{\mu\nu}$ is the transverse metric and n_λ is the normal of $\partial\mathcal{M}$. The induced metric is $h_{ij}e^i_\alpha e^j_\beta$ where $e^i_\alpha = \left(\frac{\partial x^i}{\partial x^\alpha}\right)_{\partial\mathcal{M}}$ and $n^\alpha n_\alpha = \epsilon$ [22]. Now, putting this all together, one finds the new total action:

$$\delta\Gamma^\lambda{}_{\nu\mu} = \frac{1}{2}g^{\beta\lambda}\left(\partial_\nu\delta g_{\beta\mu} + \partial_\mu\delta g_{\beta\nu} - \partial_\beta\delta g_{\nu\mu}\right) + \frac{1}{2}\delta\left(T^\lambda{}_{\nu\mu} - T_\mu{}^\lambda{}_\nu + T_{\nu\mu}{}^\lambda\right) \tag{1.62}$$

$$\delta\Gamma^\nu{}_{\nu\mu} = \frac{1}{2}g^{\beta\nu}\left(\partial_\mu\delta g_{\beta\nu}\right) + \delta T^\nu{}_{\mu\nu} \tag{1.63}$$

$$n_\lambda g^{\mu\nu}\delta\Gamma^\lambda{}_{\nu\mu} = \frac{n^\beta}{2}\left(h^{\mu\nu} + \epsilon n^\mu n^\nu\right)\left(2\partial_\nu\delta g_{\beta\mu} - \partial_\beta\delta g_{\nu\mu}\right) \tag{1.64}$$

Here I have used $g^{\alpha\beta} = h^{\alpha\beta} + \epsilon n^\alpha n^\beta$, $\delta g_{\alpha\beta}$ vanishes on $\partial\mathcal{M}$ and the tangential derivative also vanishes $\partial_\beta\delta g_{\mu\nu}e^\beta_i = 0$.

$$\begin{aligned}n_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda{}_{\nu\mu} - g^{\mu\lambda}\delta\Gamma^\nu{}_{\nu\mu}\right)\big|_{\partial\mathcal{M}} &= -\frac{n^\beta}{2}h^{\mu\nu}\left(\partial_\beta\delta g_{\mu\nu}\right) + 2h^{\beta\nu}n^\mu\delta T_{\mu\beta\nu} && (1.65)\\ &= -\delta\left(2h^{\alpha\beta}\nabla_\beta n_\alpha\right)\end{aligned}$$

Here, there is a subtlety that has to be accounted for. Since the boundary of a manifold is a submanifold, one would like to define the normal vectors of the submanifold. To define the extrinsic curvature of a submanifold, one requires the notion of a tangent space and the Weingarten map. As we'll see, this causes problems for foliating spacetimes with torsion.

Definition (Tangent and dual space [45]): Every point p on a manifold has a corresponding **tangent space**, T_pM, where the tangent space and manifold have the same dimensions. The **vectors** inside this space are the directional derivatives of smooth functions evaluated at every point on the manifold. The basis of T_pM are partial derivatives $\{\boldsymbol{\partial}/\boldsymbol{\partial x^{\rho}}\}$, where x^{ν} are the local coordinates. The dual space T_p^*M is a vector space of functions that map elements of the tangent to the real numbers. The dual space has the same dimensions as the manifold and has elements called **1-forms**. Differentials, $\boldsymbol{d}x^{\nu}$ form the basis of the dual space. If $\boldsymbol{s} \in T^*pM$ acts on $\boldsymbol{v} \in T_pM$ this is the same as $\langle \boldsymbol{s}, \boldsymbol{v} \rangle \in \mathbf{R}$, where $\langle \boldsymbol{s}, \boldsymbol{v} \rangle = s_{\nu} v^{\mu} \langle \boldsymbol{d}x^{\nu}, \boldsymbol{\partial_{\mu}} \rangle$ and $\langle \boldsymbol{d}x^{\nu}, \boldsymbol{\partial_{\mu}} \rangle = \delta^{\nu}_{\mu}$.

Definition (Weingarten map and extrinsic curvature [45]): The Weingarten map χ describes the change of the normal vector along $\partial\mathcal{M}$:

$$\begin{aligned} \chi : \mathcal{T}_p(\partial\mathcal{M}) &\longrightarrow \mathcal{T}_p(\partial\mathcal{M}) \\ \boldsymbol{v} &\longmapsto \nabla_v \boldsymbol{n} \end{aligned} \tag{1.66}$$

As you can see, the mapping is self-adjoint. Meaning $\langle \boldsymbol{u}, \chi(\boldsymbol{v}) \rangle = \langle \chi(\boldsymbol{u}), \boldsymbol{v} \rangle$ for all $\boldsymbol{u}, \boldsymbol{v}$ in the tangent space $\partial\mathcal{M}$.

The extrinsic curvature, $\boldsymbol{K}$ directly follows:

$$\boldsymbol{K} : \quad \mathcal{T}_p(\Sigma) \times \mathcal{T}_p(\Sigma) \longrightarrow \mathbf{R} \tag{1.67}$$

$$(\boldsymbol{u}, \boldsymbol{v}) \longmapsto \langle \boldsymbol{u}, \boldsymbol{\chi}(\boldsymbol{v}) \rangle \tag{1.68}$$

Here I'd like to define torsion w.r.t to vectors $\boldsymbol{x}, \boldsymbol{y}$ in the tangent space [45]:

$$\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\nabla_x y} - \boldsymbol{\nabla_y x} - [\boldsymbol{xy}] \tag{1.69}$$

Where $[\boldsymbol{x}, \boldsymbol{y}]$ is the lie derivative of $\boldsymbol{y}$ in the direction $\boldsymbol{x}$.

Consider a 3 + 1 decomposition of a Lorentizian manifold with a connection endowed with torsion. Where $\mathcal{M} = \bigcup_{t \in \mathbf{R}} \mathcal{S}_t$ [22]. From the definition above, the Weingarten map may no longer be an endomorphism for this type of foliation [13].

$$\langle \boldsymbol{x}, \boldsymbol{\chi}(\boldsymbol{y})\rangle = \boldsymbol{\nabla}_{\boldsymbol{x}}(\underbrace{\langle \boldsymbol{y}, \boldsymbol{n}\rangle}_{=0}) - \langle \boldsymbol{n}, \boldsymbol{\nabla}_{\boldsymbol{y}}\boldsymbol{x}\rangle \tag{1.70}$$

$$= -\langle \boldsymbol{n}, (\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{y} - \boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}) - [\boldsymbol{x}, \boldsymbol{y}])\rangle \tag{1.71}$$

Taking $\boldsymbol{n} = \boldsymbol{d}t$, such that $t \in \mathfrak{X}(\mathcal{M})$ [13]

$$\langle \boldsymbol{d}t, [\boldsymbol{x}, \boldsymbol{y}]\rangle = \langle \boldsymbol{d}x^{\alpha}, \boldsymbol{\partial_{\beta}}\rangle \, \partial_{\alpha} t \left(x^{\rho}\partial_{\rho} y^{\beta} - y^{\rho}\partial_{\rho} x^{\beta}\right) \tag{1.72}$$

$$= \partial_{\beta} t \left(x^{\rho}\partial_{\rho} y^{\beta} - y^{\rho}\partial_{\rho} x^{\beta}\right) \tag{1.73}$$

$$= x^{\rho} y^{\beta} \left(\partial_{\beta}\partial_{\rho} t - \partial_{\rho}\partial_{\beta} t\right) = 0 \tag{1.74}$$

However $\langle \boldsymbol{n}, \boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y})\rangle$ isn't necessarily 0. Hence, the Weingarten may no longer self-adjoint. Via Frobeniuses first theorem [46][45], the distribution $\langle \boldsymbol{t}\rangle^{\perp}$ is non-integral, which means you may not be able to define a regular foliation structure for a connection endowed with torsion on a manifold foliated by time. As one can tell, torsion is no trivial object, to the point where standard formalisms become far more convoluted. There are instances where these boundary terms, including torsion, are no longer differentiable [5]; however, further discussion is beyond the scope of this dissertation.

$$\boldsymbol{S_{TOTAL}} = \int_{\mathcal{M}} R_{\mu\nu}\boldsymbol{g}^{\mu\nu} d^4x + 2\oint_{\partial\mathcal{M}} \mathrm{d}^3 y\sqrt{|h|}\boldsymbol{K} \tag{1.75}$$

Where $\boldsymbol{K}$ is the trace of the extrinsic curvature. One finds the following equations when varying the new action:

$$\frac{\boldsymbol{\delta S}}{\boldsymbol{\delta \Gamma}} = -\partial_{\beta} g^{\alpha\phi} - \Gamma^{\phi}_{\beta\mu} g^{\mu\alpha} - \Gamma^{\alpha}_{\nu\beta} g^{\nu\phi} + \delta^{\alpha}{}_{\beta}\left(\Gamma^{\phi}_{\nu\mu} g^{\nu\mu} + \partial_{\nu} g^{\phi\nu}\right) + \Gamma^{\rho}_{\rho\beta} g^{\phi\alpha} = 0 \tag{1.76}$$

Another slight error in Einstien's work. The volume element must be diffeomorphism invariant; hence, the equations taken from the action cannot be in terms of the tensor density.

Throughout the rest of the papers from 1925-1928, Einstein made valiant attempts to unify electromagnetism and gravity. Most attempts included inserting tensors into the metric or developing ansatz for actions. In later work, the Weitzenbock Connection was introduced. The connection admits zero curvature but nonzero torsion, opening the door to a wide field of study.

1.2.3 Boundary conditions and ADM energy in GR

The explicit reason for cancelling this boundary term can be illustrated by considering a simplified Hamiltonian formulation of GR. Most calculations follow and build on the ADM section in [30]. I will adapt the interpretation for extended Riemannian geometry on the manifold $(\mathcal{M}, g)$. Consider the case where there is no matter content, no torsion and spacetime is floated by non-intersecting leaves of constant time such that $\mathcal{M} = \bigcup_{t\in\mathbf{R}} \mathcal{S}_t$ [30]:

$$ds^2 = -N^2dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right) \tag{1.77}$$

Where N and N^i are the lapse and shift functions, respectively (shows how coordinates are related between leaves). A few concepts must be laid out before substituting values into the Einstein-Hilbert action.

Firstly the spatial covariant derivative D_j, the spatial connection γ^i_{jk} and the spatial Riemann tensor ${}^{(3)}R^i{}_{lkm}$ [13]:

$$D_jA^i = \partial_jA^i + \gamma^i_{jk}A^k \tag{1.78}$$

$$\gamma^i_{jk} = \frac{1}{2}h^{il}\left(\partial_jh_{kl} + \partial_kh_{jl} - \partial_lh_{jk}\right) \tag{1.79}$$

$${}^{(3)}R^i{}_{lkm} = \partial_k\gamma^i_{lm} - \partial_m\gamma^i_{kl} + \gamma^i_{kn}\gamma^n_{lm} - \gamma^i_{mn}\gamma^n_{kl} \tag{1.80}$$

The spatial Riemann tensor describes the intrinsic of the leaves. Additionally, the extrinsic curvature of each leaf is as follows [13]:

$$\boldsymbol{m} = \boldsymbol{\partial}_t - N^k\boldsymbol{\partial}_k \tag{1.81}$$

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{m}}h_{ij} = -2NK_{ij} \tag{1.82}$$

$$= m^\alpha\nabla_\alpha h_{ij} - h_{kj}D_iN^k - h_{ki}D_jN^k \tag{1.83}$$

$$= m^\alpha\left[\partial_\alpha h_{ij} - \Gamma^k_{\alpha j}h_{ik} - \Gamma^k_{\alpha i}h_{jk}\right] + \cdots$$

$$= \partial_th_{ij} - N^kD_k(h_{ij}) + \cdots$$

$$= \partial_th_{ij} - D_iN_j - D_jN_i \tag{1.84}$$

$$\therefore K_{ij} = \frac{1}{-2N}\left(\partial_th_{ij} - D_iN_j - D_jN_i\right) \tag{1.85}$$

Where I have used the $D_k(h_{ij}) = 0$. Finally using the Gauss-Codazzi equations [13] the full Riemann tensor is expressed in terms of the extrinsic curvature and spatial Riemann tensor [1]:

$$\sqrt{-g}R = N\sqrt{h}\left[K_{ij}K^{ij} - K^2 + {}^{(3)}R\right] - 2\partial_t\left(\sqrt{h}K\right) + 2\partial_i\left[\sqrt{h}\left(KN^i - D^iN\right)\right] \tag{1.86}$$

Hence the full action takes the form:

$$\boldsymbol{S} = \int d^4xN\sqrt{h}\left[K_{ij}K^{ij} - K^2 + {}^{(3)}R\right] + \boldsymbol{S}_{boundary} \tag{1.87}$$

Excluding the boundary term the action is in terms of N, N^i and h_{ij}. Varying the action w.r.t N^i and N give the Hamiltonian and momentum constraints respectively. Finally varying the intrinsic metric gives the evolution equation. The next step is finding the conjugate momenta to define the Hamiltonian [30]:

$$\pi^{ij} = \frac{\delta S}{\delta\partial_t h_{ij}} = \sqrt{h}(K^{ij} - Kh^{ij}) \tag{1.88}$$

Using the Legendre transform and neglecting boundary terms [30]:

$$H = \int d^3x\sqrt{h}\left(N\mathcal{H} + N^i\mathcal{H}_i\right) \tag{1.89}$$

$$\mathcal{H} = h^{-1}\pi^{ij}\pi_{ij} - {}^{(3)}R - \frac{1}{2}h^{-1}\pi^2 \tag{1.90}$$

$$\mathcal{H}_i = -2h_{ik}D_j\left(h^{-\frac{1}{2}}\pi^{jk}\right) \tag{1.91}$$

$$\pi = h^{ij}\pi_{ij} \tag{1.92}$$

The Hamiltonian formalism has h_{ij} and π^{ij} as dynamic, implying $\frac{\delta H}{\delta N} = \frac{\delta H}{\delta N^i} = 0$, hence $\mathcal{H} = \mathcal{H}_i = 0$ [30]. The equations of motion are [30]:

$$\partial_t h_{ij} = \frac{\delta H}{\delta\pi^{ij}} \tag{1.93}$$

$$\partial_t\pi^{ij} = -\frac{\delta H}{\delta h_{ij}} \tag{1.94}$$

An immediate problem arises; the Hamiltonian vanishes! The resolution to this lifeless universe is to add boundary terms. Boundary terms are neglected when calculating 1.93 and 1.94. These terms would indeed generate boundary terms. If constant leaves are *compact* then no surface term would exist, hence in a closed universe, the Hamiltonian is indeed zero, even in the presence of matter [30]. Consider the case where this is not the case and the leaves are asymptotically flat. One can evaluate boundary terms in the limit of $r \to \infty$. Since the leaves are asymptotically flat $h_{ij} = \delta_{ij} + \mathcal{O}\left(\frac{1}{r}\right)$ and $\pi^{ij} = \mathcal{O}\left(\frac{1}{r^2}\right)$. Hence, $\delta h_{ij} = \mathcal{O}\left(\frac{1}{r}\right)$ and $\delta\pi^{ij} = \mathcal{O}\left(\frac{1}{r^2}\right)$. Additionally, assume $N = 1 + \mathcal{O}\left(\frac{1}{r}\right)$ and $N^i \to 0$ in the limit of $r \to \infty$ [31]. When varying π^{ij}, the surface term, defined as an S^2 sphere of constant r is [30]:

$$\int_{S^2} dA \left(-2N^i h_{ik} n_j \frac{1}{\sqrt{h}} \delta\pi^{jk}\right) \tag{1.95}$$

Where n_j is the unit normal and dA is the area element. Now taking the limit as $r \to \infty$:

$$h_{ij} \sim \delta_{ij} + \frac{1}{r},\ dA \sim r^2,\ \frac{1}{\sqrt{h}} \sim \sqrt{r},\ \delta\pi^{ij} \sim \frac{1}{r^2},\ N^i \sim \frac{1}{r} \tag{1.96}$$

$$\lim_{r\to\infty}\left(\int_{S^2} dA \left(-2N^i h_{ik} n_j \frac{1}{\sqrt{h}} \delta\pi^{jk}\right)\right) \sim r^2 \frac{1}{r}\left(\frac{1}{r} + \delta_{ij}\right)\sqrt{r}\frac{1}{r^2} = 0 \tag{1.97}$$

Hence the boundary term for π^{ij} has no effect. However, when varying h_{ij} this is no longer the case. Two surface terms arise from these variations. Firstly from varying $\frac{1}{\sqrt{h}}$ in $\mathcal{H}_i$.

$$H = \int dx^3 \sqrt{h}\left(N^i \mathcal{H}_i + \cdots\right) \tag{1.98}$$

$$\delta H[\boldsymbol{h}, \boldsymbol{\pi}; \boldsymbol{\delta det(h)}] = \int dx^3 \sqrt{h}\left[-2h_{ik} N^i D_j \left(\delta h^{-\frac{1}{2}} \pi^{jk}\right)\right] + \cdots \tag{1.99}$$

$$\delta h = h h^{lm} \delta h_{lm} \tag{1.100}$$

$$\therefore \delta H[\boldsymbol{h}, \boldsymbol{\pi}; \boldsymbol{\delta det(h)}] = \int dx^3 \sqrt{h}\left[-2h_{ik} N^i D_j \left(\delta h^{-\frac{1}{2}} \pi^{jk}\right)\right] + \cdots \tag{1.101}$$

$$= \int dx^3 \sqrt{h}\left[D_j \left(h_{ik} N^i h^{lm} \delta h_{lm} h^{-1/2} \pi^{jk}\right)\right] + \cdots \tag{1.102}$$

$$= \int_{S^2} dA \left(n_j h_{ik} N^i h^{lm} \delta h_{lm} h^{-1/2} \pi^{jk}\right) + \cdots \tag{1.103}$$

Taking the limit:

$$\lim_{r\to\infty}\left(\int_{S^2} dA\left(n_j h_{ik} N^i h^{lm}\delta h_{lm} h^{-1/2}\pi^{jk}\right)\right) \tag{1.104}$$
$$\sim \lim_{r\to\infty}\left(r^2\left(\delta_{ik}+\frac{1}{r}\right)\frac{1}{r}\left(\delta^{lm}-\frac{1}{r}\right)\frac{1}{r}\sqrt{r}\frac{1}{r^2}\right)=0$$

Where I have used that @ $\mathcal{O}\left(\frac{1}{r}\right)$ $h^{ij}h_{ij}=3 \Rightarrow h^{ij}=\delta^{ij}-\mathcal{O}\left(\frac{1}{r}\right)$. Hence this boundary term doesn't contribute. Finally, the second boundary term comes from varying ${}^{(3)}R$. A nice approach I haven't seen is linearising the spatial Riemann tensor before the calculation. Since the space is asymptotically flat this is a reasonable procedure to do in the limit of $r\to\infty$:

$$h_{ij}=\delta_{ij}+\delta h_{ij} \tag{1.105}$$

$$\delta^{(3)}R_{ijkl}=\frac{1}{2}\left[\partial_j\partial_k\delta h_{il}+\partial_i\partial_l\delta h_{jk}-\partial_j\partial_l\delta h_{ik}-\partial_i\partial_k\delta h_{lj}\right]+\mathcal{O}\left(\delta h_{ij}^2\right) \tag{1.106}$$

$$\delta^{(3)}R=\partial_i\partial_j\delta h_{ij}-\partial^2\delta h_{jj} \tag{1.107}$$

Plugging this into the Hamiltonian reveals:

$$\delta H\left[\boldsymbol{\delta h};\boldsymbol{h},\boldsymbol{N},\boldsymbol{N^i}\right]=\int -N\delta^{(3)}R+\cdots\sqrt{h}d^3x \tag{1.108}$$

$$=\int -N\left(\partial_i\partial_j\delta h_{ij}-\partial^2\delta h_{jj}\right)+\cdots\sqrt{h}d^3x \tag{1.109}$$

$$\tag{1.110}$$

Integrating by parts reveals a boundary term [30]:

$$\delta H\left[\boldsymbol{\delta h};\boldsymbol{h},\boldsymbol{N},\boldsymbol{N^i}\right]=-\int_{S^2} dA\; Nn_i\left(\partial_j\delta h_{ij}-\partial_i\delta h_{jj}\right) \tag{1.111}$$

$$N\sim 1+\frac{1}{r} \tag{1.112}$$

$$\therefore \lim_{r\to\infty}\left(\delta H\left[\boldsymbol{\delta h};\boldsymbol{h},\boldsymbol{N},\boldsymbol{N^i}\right]\right)\sim r^2\left(1+\frac{1}{r}\right)\frac{1}{r^2}\sim 1 \tag{1.113}$$

As you can see, this means this boundary term will not generally vanish and will contribute to the Hamiltonian. Integrating by parts reveals a second boundary term that does vanish:

$$\lim_{r\to\infty}\left(\int_{S^2} dA\; n^j\left(\delta h_{ij}\partial^i N-\delta h_{kk}\partial_j N\right)\right) \tag{1.114}$$
$$\sim \lim_{r\to\infty} r^2\frac{1}{r}\frac{1}{r^2}=0$$

The non-vanishing boundary term is referred to commonly in literature as the ADM energy for an asymptotically flat end [30]:

$$\lim_{r\to\infty}\left(-\int_{S^2} dANn_i\left(\partial_j\delta h_{ij}-\partial_i\delta h_{jj}\right)\right)=\delta E_{ADM} \tag{1.115}$$

$$\tag{1.116}$$

where

$$E_{ADM}=\lim_{r\to\infty}\int_{S^2} dAn_i\left(\partial_j h_{ij}-\partial_i h_{jj}\right) \tag{1.117}$$

The prescription is to consider a modified Hamiltonian [30]:

$$H^{'}=H+E_{ADM} \tag{1.118}$$

$H^{'}$ and H give the same equations of motion, however with the E_{ADM} term in $H^{'}$ the leaves are no longer compact. This is because E_{ADM}, in effect, cancels out the non-zero boundary term when varying h_{ij}. Hence the true Hamiltonian for an asymptotically flat space in GR is $H^{'}$ [30]. Regge and Teitelboim [31] first discovered the need for a surface term in 1974 for the Hamiltonian formalism of GR. The reasoning is extended to the GHY boundary term formulated earlier on. This is done to ensure consistency when calculating the boundary of a black hole. The reason for showing a simplified example is to illustrate the importance of boundary terms even in the context of the 3 + 1 decomposition of spacetime.

1.3 Fibre bundles and Cartan's structure equation

1.3.1 Fibre Bundles

Let's outline some crucial concepts before coupling spinors to gravity. The language of fibre bundles provides a rigorous mathematical structure for gauge field theories on general differential manifolds. The most notable gauge theory in electromagnetism is Yang-Mills. The theory impressively unifies the weak and strong forces. The definitions in this section closely follow *Geometry, topology and physics* by Nakahara [26], Frederic Schuller's lecture series on the Geometrical Anatomy of Theoretical Physics [33] and for mathematical rigour Topology, Geometry and Gauge fields by Gregory L.Naber [25].

Definition (Principle bundles [25] [26]): A coordinate bundle consists of several components: A total space P and a base manifold M with a surjective map π that takes points from the total space and projects them down to the base manifold. The inverse map $\pi^{-1}(l), l\in M$ (**canonical**

trivialisation, $s = \pi^{-1}$) defines the set points on P known as the fibre over l.

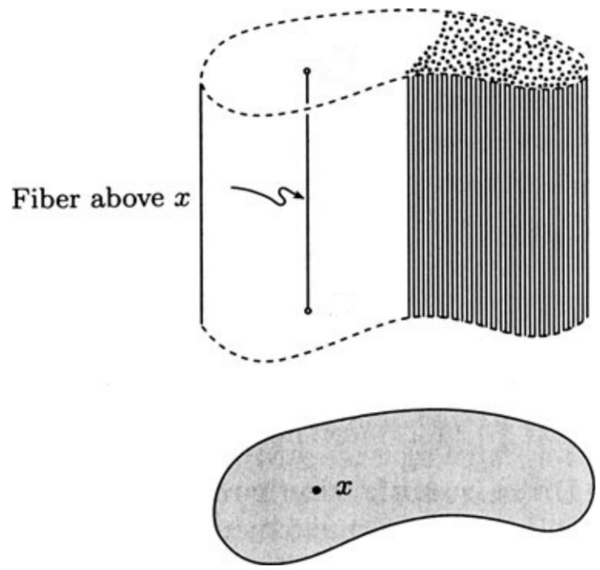

Figure 1.11: [25] The following is a depiction of a fibre F over a point $x \in M$

Say there's a non-empty intersection of open coverings on the base manifold, $U_i \cap U_j \neq \emptyset$. How do you get from fibres described over U_i to fibres described in U_j?
In the same way, differentiable manifolds look locally like $\mathbb{R}^n$; the total space locally looks like $U \times G$. Where G is the structure group of the principle bundle. Much like a coordinate, the local trivialisation takes a point from the total space to $U \times G$, via $\Psi\,(\psi, U)$, where $\psi : s(U) \to U \times G$.

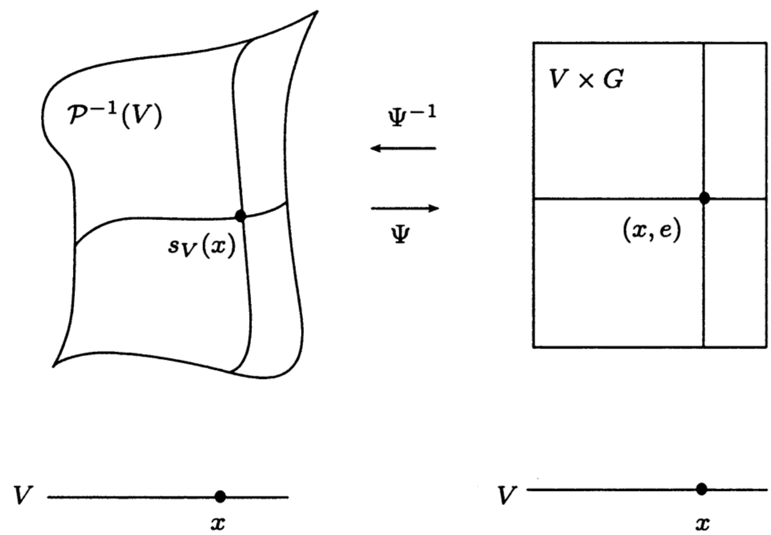

Figure 1.12: [25] The following depicts a map from the total space to the local trivialisation. Note there are a few changes in notation. Notably $\mathcal{P}$ is the surjective map π and V is the open covering on the base manifold instead of U

We now have the toolkit to define the transition function $g_{ji}(l) := \psi_j(l) \circ \psi_i^{-1}(l)$ which tasks you from different open covers, $s_j = g_{ji}s_i$. The following diagram illustrates precisely what is going on:

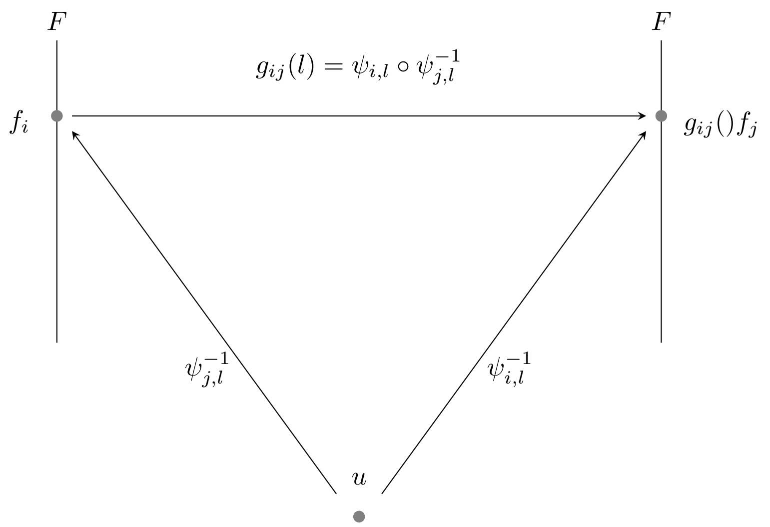

Figure 1.13: [26] Illustration of transition function $g_{ij}(l) : G \to G$ acting on the fibre F, where the point $u \in \pi^{-1}(l) \times G$

In the same way that the equivalence class of atlases form a differentiable manifold, the equivalence class of coordinate bundles defines a principle bundle. Note: this is not the most general bundle; the principle bundle is a fibre bundle with fibres $F \in G$. The principle is usually denoted as P(M,G).

A few more tedious definitions are required to know precisely what is going on from the geometric perspective when talking about gauge theories. To understand how to gauge potentials look geometrically a few more notions need to be solidified: vector bundle, cotangent bundle and frames.

Definition (Vector bundle [26]): When the fibre, F of a fibre bundle (E, π, M, F, G) is a vector space, the fibre bundle is known as a **vector bundle.** Let $F = \mathbb{R}^k/\mathbb{C}^k$, where k is the **fibre dimension** and $dim\,(M) = m$. The transition function is characterised by $GL\left(k, \mathbb{R}^k/\mathbb{C}^k\right)$.

Definition (Tangent/Co-tangent bundle [26]): A tangent bundle TM is the union of all the tangent spaces of the base space M. This is denoted as follows:

$$TM = \bigcup_{p \in M} T_pM \tag{1.119}$$

An open covering $\{U_i\}$ of M with the coordinate $x^\rho = \psi_i(p)$. The total space of the coordinate bundle is denoted as:

$$TU_i = \bigcup_{p \in U_i} T_p M \tag{1.120}$$

TU_i is the direct product $U_i \times \mathbf{R}^m$ and mapping for a local piece $\pi : TU_i \to U_i$, where $\pi^{-1}(p) = (p, V)$. The vector V is an element of the tangent space evaluated at point p s.t $V = V^\rho(p)\boldsymbol{\partial_\rho}$. Taking a local section of U_i on M a nice visualisation of what π is doing. One could imagine a piston squashing the total space onto the base space leaving an imprint on the base space.

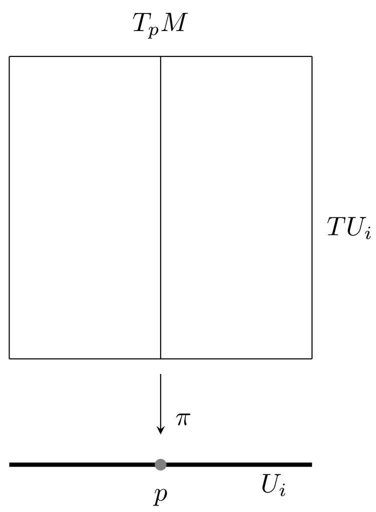

Figure 1.14: Visualisation of a local piece of the total space, TU_i projecting a vector $V \in T_pM$ to p [26]

For a tangent bundle, the structure group takes an interesting form. Consider the intersection of two charts, $p \in U_i \cap U_j$. The vector V now has two different coordinate representations $(U_i, x^\beta), (U_j, y^\beta)$ s.t:

$$\boldsymbol{V} = V^\beta \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^\beta}}\bigg|_p = \bar{V}^\beta \frac{\boldsymbol{\partial}}{\boldsymbol{\partial y^\beta}}\bigg|_p \tag{1.121}$$

The relation between the two representations is:

$$\bar{V}^\beta = \frac{\partial x^\beta}{\partial y^\mu} V^\mu = G^\beta{}_\mu V^\mu \tag{1.122}$$

Where the matrix $G^\beta{}_\mu \in GL(m, \mathbf{R})$ is non-singular. This matrix represents the structure group of a tangent bundle. It can be thought of as a rotation of the fibre coordinates. The definition of a Cotangent/dual bundle is very similar to the tangent bundle except for the basis of the cotangent space T_p^*M with chart $(U_i, x^\mu))$ has the basis $\{\boldsymbol{d}x^\mu\}$. The cotangent bundle is denoted as:

$$T^*M = \bigcup_{p \in M} T_p^*M \tag{1.123}$$

For an intersection of open coverings $U_i \cap U_j$, the transformation between basis is as follows:

$$\boldsymbol{d}y^\mu = \boldsymbol{d}x^\nu \frac{\partial y^\mu}{\partial x^\nu}\bigg|_p \tag{1.124}$$

$$\boldsymbol{\omega} = \omega_\mu \boldsymbol{d}x^\mu = \bar{\omega}_\mu \boldsymbol{d}y^\mu \tag{1.125}$$

$$\omega_\mu = \bar{\omega}_\nu \frac{\partial y^\mu}{\partial x^\nu}\bigg|_p = G_\mu{}^\nu(p)\bar{\omega}_\nu \tag{1.126}$$

In this case however $G(p)$ is the transition function $g_{ji}(p)$.

Definition (Frames[26]): Each fibre of a tangent bundle has a basis $\{\boldsymbol{\partial_\mu}\}$ given by the altas (U_i, x^μ). If M has a metric, then the same orthonormal basis $\{\boldsymbol{e^\mu}\}$ is a vector field on U_i. They define the **local frame** on U_i.

Finally, there is a small discussion about associated bundles.

Definition (Associated bundles [26]): One can construct an associated bundle from a principle bundle, $P\,(M, G)$.
G acts on the left of a fibre, F of the associated fibre bundle, (E, π, M, G, F, P). The action $g \in G$ on $P \times F$ is:

$$(u, f) \to \left(ug, g^{-1}f\right) \tag{1.127}$$

For the associated bundle the map π_F from the total space to base space is less trivial than the bog standard fibre bundle definition. Firstly the total space of the associated fibre bundle is the equivalence class of the quotient group:

$$E = P \times \frac{M}{G} \tag{1.128}$$

$$\pi_F : E \to M \tag{1.129}$$

$$\pi_F(u, v) = \pi(u) \tag{1.130}$$

Where $\pi(u)$ is the map from the total space, P of the principle bundle to the base space, M. π_F is still well defined as the $\pi(ug) = \pi(u)$. Note if the transition function (given a representation) for an associated bundle and principle bundle then there is a bundle isomorphism between the two. Hence if the transition function of E is $\rho\left(g_{ij}\right)$, where g_{ij} is the transition function for the principle bundle. The best way to see how an associated bundle manifests itself is through an example (Lecture 20 [33]).

Take the tangent bundle TM with an m-dimensional base space M endowed with a metric. There is an associated frame bundle $LM_{\mathbb{R}^m}$, where LM denoted as:

$$LM = \bigcup_{p \in M} L_p M \tag{1.131}$$

$$LM_{\mathbb{R}^m} = LM \times \mathbb{R}^m \tag{1.132}$$

Where L_pM is a set of frames for at p. Defining a chart (U_i, x^μ) on M, where T_pM has the basis $\{\boldsymbol{\partial_\mu}\}$ on U_i. The frame $\{X_\alpha\}$ at p is as follows:

$$\boldsymbol{X}_\alpha = X^\mu{}_\alpha \ \boldsymbol{\partial_\mu}|_p \tag{1.133}$$

Note $X^\mu{}_\alpha \in GL(m, \mathbf{R})$ so the frame has linearly independent elements. The structure group of the associated bundle is $G = GL(m, \mathbf{R})$. The right and left actions take the form for $g \in GL(m, \mathbb{R})$ and $f \in F$:

$$\left(\{\boldsymbol{X}\}, g\right) \to \{\boldsymbol{X}\}\, g \tag{1.134}$$

$$\boldsymbol{X}_\beta \to \boldsymbol{X}_\alpha g^\alpha{}_\beta \tag{1.135}$$

$$f^\alpha \to g^{-1\alpha}{}_\beta f^\beta \tag{1.136}$$

Now take the intersection of two open coverings $p \in U_i \bigcap U_j$ with coordinates x^μ and y^μ respectively one has:

$$\boldsymbol{X}_\alpha = X^\mu{}_\alpha \ \boldsymbol{\partial_\mu}\big|_p = \bar{X}^\mu{}_\alpha \ \bar{\boldsymbol{\partial}}_{\boldsymbol{\mu}}\big|_p \tag{1.137}$$

This implies that ${X^{\mu}}_{\alpha} = \frac{\partial x^{\mu}}{\partial \bar{x}^{\nu}} {\bar{X}^{\nu}}_{\alpha}$, hence the transition function is the same as the transition function for TM. From the definitions laid out above a bundle isomorphism is now able to be defined between LM and TM

$$\begin{tikzcd} LM_{\mathbb{R}^m} \arrow[r, "\eta"] \arrow[d, "\pi_F"'] & TM \arrow[d, "\pi"] \\ M \arrow[r, "id_M"] & M \end{tikzcd}$$

Figure 1.15: The following illustrates a bundle isomorphism between an associated bundle and principle bundle [33]

The frame bundle will become useful later on in formulating general relativity in the context of fibre bundles, by restricting the structure group of the frame bundle and introducing Veilbeins.

Having established the correct toolkit, one can begin by defining a connection on a principle bundle. This will show how a gauge potential transforms in the total space. The field strength tensor in Yang-Mills is a form of curvature tied to the connection. The connection gives rise to a covariant derivative in a given vector bundle.

1.3.2 Local connection and gauge potential

Following the definition of the connection on the principle bundle, one can pullback the connection to the base manifold M. Since the pulledback connection is defined only at a point on the base manifold, it's considered a "local" connection. The best analogy I can give is a physical one. Imagine a wooden board, representing the base manifold and nails representing the fibres. The projection is like hammering the nails into the wooden board, the way the "local" connection is felt on the base manifold is by running your finger across the wooden board and feeling the heads of the nails at each point.

Extending this notion to an open covering, $\{U_i\}$ on the base manifold, M with a local section s_i defined on each open set. The pulledback one form $\mathcal{A}_i$ on U_i is denoted as [25][33][26]:

$$\mathcal{A}_i : \Gamma(TU_i) \to T_e G \tag{1.138}$$

$$\mathcal{A}_i := s_i^* \omega \in \mathfrak{g} \otimes \Omega^1 (U_i) \tag{1.139}$$

Note that $\Gamma(TU_i)$ just means a section of the tangent space of U_i

$$\begin{tikzcd}
P \\
P \arrow[u, "\triangleleft G"'] \arrow[d, "\pi"] \\
U_i \arrow[u, bend left=60, "s_i"]
\end{tikzcd}$$

Figure 1.16: Shows a map diagram of the local representation of a connection [33]

Note that $\triangleleft G : P \to P \times G \to P$ [33] is denoted as the right action of G on P, this follows the description in the definition of the principle fibre bundle. $\mathcal{A}_i$ is the gauge potential we see in physics, and in Dr Frederic's lectures, they are denoted as a "*Yang-Mills field*" [33]. The local trivialisation of the principle bundle, P is pulledback to the open cover U_i to form a local representation of ω on U_i [33]:

$$\phi_i : U_i \times G \to P \tag{1.140}$$

$$\phi_i^* \omega : \Gamma \left(T \left(U_i \times G \right) \right) \to T_e G \tag{1.141}$$

The Yang-Mills field and local representation are related by [33]:

$$\left(\phi_i^* \omega \right)_u (a, b) = Ad_{g^{-1}} \left(\mathcal{A}_i(a) \right) + \Xi_g(b) \tag{1.142}$$

Where $u \in U_i \times G$ and $a \in T_{u_0} U_i$, $b \in T_{u_1} G$. The **Maurer-Cartan** form, $\Xi_g : T_g G \to T_e G$. This a lie algebra valued 1-form takes in a left-invariant vector field from a Lie algebra $L_{g*} X|_h$ and spits out $X|_h$. To visualise this let's take the manifold of the Lie group G:

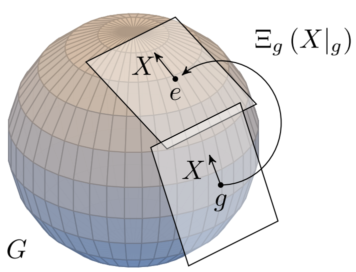

Figure 1.17: A left-invariant vector field that generates the tangent vectors from a push toward L_{g*}. The first tangent plane is $X|_e$ the second is $L_{g*} X|_e = X|_g$

Note: For simplicity, the notation will change slightly as the objects we are now dealing with are matrices so $X|_e = X(e),\ L_{g*}X|_e = X(g)$

Lets now take a look at constructing the Maurer-Cartan form for a general linear group, $GL\,(d,\mathbb{R})$ on an "open submanifold" [25] $\mathbb{R}^{d^2}$. The coordinate entry function, x^{ij}, is denoted as $x^{ij}(g) = g^{ij},\ g^{ij}$ s.t $g \in GL(d,\mathbb{R})$. Now we can construct a basis for the Maurer-Cartan form. In this case let $\{\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^{ij}}}\big|_{\boldsymbol{e}}\}$ be the basis for the lie algebra, $\mathfrak{g}$ and $\{\Xi^{ij}\}$ is the left invariant $\mathbb{R}$-valued 1 form on G. The dual basis is $\{\Xi^{ij}(e) = \boldsymbol{dx^{ij}(e)}\}$, where e is the identity element of the group G. Now take the left-invariant vector field $X \in \mathfrak{g}$ and define it using matrix notation s.t (pg 292 [25]):

$$\mathbf{X}(e) = X^{ij}\,\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^{ij}}}\bigg|_{\boldsymbol{e}} \tag{1.143}$$

$$\mathbf{X}(g) = \left(\sum_{k=1}^{n} g^{ik}X^{kj}\right)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^{ij}}}\bigg|_{\boldsymbol{g}} = \left(\sum_{k=1}^{n} x^{ik}(g)X^{kj}\right)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^{ij}}}\bigg|_{\boldsymbol{g}} \tag{1.144}$$

Applying the Maurer-Cartan to the above vector field one finds [25]:

$$\Xi(g)^{ij}(X(g)) = \sum_{h=1}^{d}(g^{-1})^{ih}\boldsymbol{dx^{hj}}(g)\left(\sum_{k=1}^{d} x^{lk}(g)X^{km}\right)\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x^{lm}}}\bigg|_{\boldsymbol{g}} \tag{1.145}$$

$$= \sum_{h=1}^{d}\sum_{k=1}^{d}(g^{-1})^{ih}g^{hk}X^{kj} \tag{1.146}$$

$$= X^{ij} \tag{1.147}$$

Hence we recover the generating vector field [25]:

$$\boldsymbol{\Xi(g)(X(g)) = X(e)} \tag{1.148}$$

The most important part is seeing how this applies to gauge theories seen in physics. More specifically what a gauge transformation means from the perspective of differential geometry.

1.3.3 Gauge maps and connection transformation

Let's say you have a principle bundle $P(M, G)$ and want to see how the local representation of the one form on P changes between charts. Constructing the following:

$$\begin{tikzcd} & P & \\ U_i \arrow[ur] \arrow[ur, bend left=60, "s_i"] & & U_j \arrow[ul] \arrow[ul, bend right=60, "s_j"'] \end{tikzcd}$$

Figure 1.18: Map diagram of how two open sets on the base manifold M relate to the principle Bundle [33]

1.18 defines two open sets on the base manifold, M with the local cross-sections s_i and s_j.

Lemma 3. *Let a Principle-bundle $P(M, G)$ and a $\mathfrak{g}$-valued one form, ω that satisfies the definition of a connection. Let the trivialisations (U_i, ψ_i) and (U_j, ψ_j) have the intersection $U_i \cap U_j \neq \emptyset$ and the transition function g_{ij}. The relation between the local connections is [25] :*

$$\mathcal{A}_j = ad_{g_{ij}^{-1}} \circ \mathcal{A}_i + \Xi_{ij} \tag{1.149}$$

Where $\Xi_{ij} = g_{ij}^{*}\Xi$. Put in more explicit terms for $p \in U_i \cap U_j$ and $\boldsymbol{v} \in T_pM$, $\mathcal{A}_j(p)(\boldsymbol{v}) = g_{ij}(p)^{-1}\mathcal{A}_i(p)(\boldsymbol{v})g_{ij}(p) + \Xi_{g_{ij}(p)}(g_{ij*}\boldsymbol{v}(p))$ [25]

Now defining the Maurer-Cartan form the way physicists usually see it.

Let α be a curve in the base manifold M and $\alpha'(0) = \mathbf{v}$ then [25]:

$$\Xi_{g_{ij}(p)}\left((g_{ij})_{*p}(\mathbf{v})\right) = \Xi_{g_{ij}(p)}\left((g_{ij})_{*p}(\alpha'(0))\right) \tag{1.150}$$

$$= \Xi_{g_{ij}(p)}\left((g_{ij} \circ \alpha)'(0)\right) \tag{1.151}$$

$$= \left(L_{(g_{ij}(p))^{-1}}\right)_{*g_{ij}(p)}\left((g_{ij} \circ \alpha)'(0)\right) \tag{1.152}$$

$$= \left(L_{(g_{ij}(p))^{-1}} \circ g_{ij} \circ \alpha\right)'(0) \tag{1.153}$$

$$= \frac{d}{dt}\left[(g_{ij}(p))^{-1}(g_{ij} \circ \alpha)(t)\right]\Bigg|_{t=0} \tag{1.154}$$

$$= (g_{ij}(p))^{-1}(g_{ij} \circ \alpha)'(0) \tag{1.155}$$

$$= (g_{ij}(p))^{-1} dg_{ij}(p)(\mathbf{v}) \tag{1.156}$$

Looking back, a local $U(1)$ transformation is clearly defined in this scheme. Since the $g_{12}(p) \in U(1)$ for $p \in U_1 \cap U_2$. Note to do this one must define the chart (V, x) over the base manifold s.t $V \subseteq U_2 \cap U_1$, this ensures the pullback one-forms on the base manifold have the same basis [26]:

$$g_{12}(p) = e^{i\lambda(p)} \tag{1.157}$$

$$\therefore \mathcal{A}_{2\mu} = e^{-i\lambda(p)} \mathcal{A}_{1\mu} e^{-i\lambda(p)} + i\partial_\mu \lambda(p) \tag{1.158}$$

$$= \mathcal{A}_{1\mu} + i\partial_\mu \lambda \tag{1.159}$$

Where the components of the one forms were taken for succinctness. **Note:** The normal gauge potential in physics is $\mathcal{A}_k = iA_k$.

Consider again the frame bundle LM over M. When defining a chart (U_i, x_i) on a smooth manifold, this automatically induces a local section $s_i : U_i \to LM$ s.t (Lecture 22 [33]):

$$s_i(p) := \left(\frac{\partial}{\partial x_i^1}, \cdots, \frac{\partial}{\partial x_i^m} \right) \in L_p M \tag{1.160}$$

Now defining two charts (U_1, x) , (U_2, y) s.t $U_1 \cap U_2 \neq \emptyset$ one finds an interesting transformation of the gauge potentials (Lecture 22 [33]):

$$(\mathcal{A}_2)^i{}_{j\mu} \boldsymbol{dx^\mu} = \left(g_{12}(p)^{-1}\right)^i{}_k (\mathcal{A}_1)^k{}_{l\nu} \left(g_{12}(p)\right)^l{}_j \boldsymbol{dy^\nu} + \left(g_{12}(p)^{-1}\right)^i{}_k \partial_\nu \left(g_{12}(p)\right)^k{}_j \boldsymbol{dy^\nu} \tag{1.161}$$

$$\left(g_{12}(p)\right)^i{}_j = \frac{\partial y^i}{\partial x^j} \tag{1.162}$$

$$\Rightarrow (A_2)^i{}_{j\mu} = \frac{\partial y^\nu}{\partial x^\mu} \left(\frac{\partial x^i}{\partial y^k} (\mathcal{A}_1)^k{}_{l\nu} \frac{\partial y^l}{\partial x^j} + \frac{\partial x^i}{\partial y^k} \frac{\partial^2 y^k}{\partial x^\nu \partial x^j} \right) \tag{1.163}$$

One notes something quite striking that this connection transforms in a similar manner to the Levi-Civita connection. The one-form connection also has a notion of curvature known as the field strength tensor, much like the Riemann curvature tensor the one-form connection has the two-form curvature, $\boldsymbol{\Omega}$. However, the indices i, j are lie algebra indices of the general linear group.

1.3.4 Curvature

The curvature two-form is defined via the covariant exterior derivative and is characterised by the **Cartan structure** equation. Taking the definition of the connection on a principle bundle $P(M,G)$ endowed with a connection $\boldsymbol{\omega}$ [25]:

$$\boldsymbol{\Omega}(p)(\mathbf{v},\mathbf{w}) = (\boldsymbol{\mathcal{D}}\boldsymbol{\omega})_p\,(\mathbf{v},\mathbf{w}) \tag{1.164}$$

$$= (d\boldsymbol{\omega})_p(\mathbf{v},\mathbf{w}) + [\boldsymbol{\omega}_p(\mathbf{v}),\boldsymbol{\omega}_p(\mathbf{w})] \tag{1.165}$$

The full derivation of the covariant derivative is lengthy but straightforward. It involves splitting up the tangent vector space at each point of the total space into two horizontal and vertical subspaces. From this, the covariant derivative only acts on the horizontal subspace. Additionally, left and right group actions on the connections reveal the transformation properties between open coverings of the base manifold. The full derivation can be seen in [26] and [25]. Now written more succinctly, the Cartan structure equation takes the form [25]:

$$\boldsymbol{\Omega} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}] \tag{1.166}$$

Note: $[\boldsymbol{\omega},\boldsymbol{\omega}]$ is the wedge product determined by the Lie bracket in $\mathfrak{g}$, so $[\boldsymbol{\omega},\boldsymbol{\omega}]_p(\mathbf{v},\mathbf{w}) = [\boldsymbol{\omega}_p(\mathbf{v}),\boldsymbol{\omega}_p(\mathbf{w})] - [\boldsymbol{\omega}_p(\mathbf{w}),\boldsymbol{\omega}_p(\mathbf{v})] = 2\,[\boldsymbol{\omega}_p(\mathbf{v}),\boldsymbol{\omega}_p(\mathbf{w})]$ [25].
The right action, σ on the curvature is [25]:

$$\sigma_g^*\boldsymbol{\Omega} = \sigma_g^*\left(d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}]\right) \tag{1.167}$$

$$= \sigma_g^*(d\boldsymbol{\omega}) + \frac{1}{2}\sigma_g^*([\boldsymbol{\omega},\boldsymbol{\omega}]) \tag{1.168}$$

$$= d\left(ad_{g^{-1}} \circ \boldsymbol{\omega}\right) + \frac{1}{2}([\sigma_g^*\boldsymbol{\omega},\sigma_g^*\boldsymbol{\omega}]) \tag{1.169}$$

$$= ad_{g^{-1}} \circ \left(d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega},\boldsymbol{\omega}]\right) \tag{1.170}$$

$$= ad_{g^{-1}} \circ \boldsymbol{\Omega} \tag{1.171}$$

Additionally, the curvature is skew symmetry and bilinear. As we'll see this has implications on the **local**

field strength tensor.

The local field strength tensor is the pullback of the curvature two-form by the same local section, $s : U \to P$, as the gauge potential [25]:

$$\begin{aligned}\boldsymbol{\mathcal{F}} &= s^*\boldsymbol{\Omega} &(1.172)\\ &= d\left(s^*\boldsymbol{\omega}\right) + \frac{1}{2}\left[s^*\boldsymbol{\omega}, s^*\boldsymbol{\omega}\right] &(1.173)\\ &= d\boldsymbol{\mathcal{A}} + \frac{1}{2}[\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{A}}] &(1.174)\end{aligned}$$

Following similar motions as 1.149 for $p \in U_i \cap U_j$ and $\mathbf{v}, \mathbf{w} \in T_pM$, with the sections $s_i : U_i \to \pi^{-1}(U_i)$ and $s_j : U_j \to \pi^{-1}(U_j)$ then [25]:

$$\left({s_j}^*\boldsymbol{\Omega}\right)_p(\mathbf{v}, \mathbf{w}) = \boldsymbol{\Omega}_{s_j(p)}\Big(\left(s_j\right)_{*p}(\mathbf{v}), \left(s_j\right)_{*p}(\mathbf{w})\Big) \tag{1.175}$$

$$\left({s_j}^*\boldsymbol{\Omega}\right)_p(\mathbf{v}, \mathbf{w}) = g_{ij}(p)^{-1}\left({s_i}^*\boldsymbol{\Omega}\right)_p(\mathbf{v}, \mathbf{w}) g_{ij}(p) \tag{1.176}$$

$$\Rightarrow \left(\boldsymbol{\mathcal{F}}_j\right)_p = g_{ij}(p)^{-1}\left(\boldsymbol{\mathcal{F}}_i\right)_p g_{ij}(p) \tag{1.177}$$

The Riemann curvature admits the same structure as the field strength two-form:

$$\boldsymbol{R} = \boldsymbol{d\Gamma} + \frac{1}{2}\left[\boldsymbol{\Gamma}, \boldsymbol{\Gamma}\right] \tag{1.178}$$

This can be constructed by having a frame bundle with the right action group, $O(m)$, as seen in the definition of the frame bundle. As we'll see when picking the correct structure group, the local connections admit enough structure to contain the Riemann curvature tensor plus some other values.

1.3.5 Bianchi Identity

Before moving on to the next section, there is a very useful identity when taking the covariant derivative of the curvature, the **Bianchi Identity** [26]:

$$\left(\boldsymbol{\mathcal{D}}\boldsymbol{\Omega}\right)_p(\mathbf{v}, \mathbf{w}, \mathbf{x}) = 0 \tag{1.179}$$

1.3.6 Torsion revisited

Torsion on a principle bundle is introduced via soldering one-form, $\boldsymbol{\theta} \in \Omega^1(P) \otimes V$. Instead of the one form being a $\mathfrak{g}$ valued entity, it is a V valued entity, where V is the representation space of the structure group, G. The conditions the soldering form must follow are [33]:

- V is a linear representation space of G, s.t $dim(V) = dim(M)$
- $\forall X \in \Gamma(TP), \quad \boldsymbol{\theta}\left(X^V\right) = 0$
- $L_g \circ (R_g)^* \boldsymbol{\theta} = \boldsymbol{\theta}$
- The solder form has values in a linear representation V of G such that an associated vector bundle $P \times_G V$ to the tangent bundle, TM is a bundle isomorphism.

The torsion form $\boldsymbol{\Theta}$ is defined as the exterior covariant derivative of solder from s.t:

$$\begin{aligned} \boldsymbol{\Theta} &= \boldsymbol{\mathcal{D}\theta} && (1.180) \\ &= \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\theta}] + \boldsymbol{d\theta} && (1.181) \end{aligned}$$

Since the solder form is V values a gauge transformation occurs in the defining representation s.t:

$$\boldsymbol{\theta_j}(p)(\mathbf{v}) = g_{ij}(p)^{-1} \boldsymbol{\theta_i}(p)(\mathbf{v}) \tag{1.182}$$

Where the point $p \in V_i \cap V_j$ and (V_j, ϕ_j), (V_i, ϕ_i) are local trivialisation's in a principle bundle. Pulling back the solder form to the base manifold yields the torsion tensor where $s^* \boldsymbol{\Theta} = \boldsymbol{T}$.

1.3.7 General relativity reformulated

Restricting the structure group of the frame bundle to the lie algebra $SO(3,1)$ produces the correct structure for general relativity.

Theorem 4 (Connection reduciblility). *"Given a fibre metric* $\boldsymbol{g}$ *of an associate vector bundle* $(E, \boldsymbol{F^m}, G, P)$ *and* $Q(M, H)$ *the reduced sub bundle of the fibre bundle* $P(M, G)$*. A connection,* Γ *in* P *is reducible to a connection in* Q *if and only if* Γ *is a metric connection." [21]*

The principle fibre bundle, $P(M, SO(3,1))$, where M is endowed with a metric, g is a sub-bundle of the frame bundle LM. In fibre bundle language g is a global section of a $(0,2)$ tensor bundle over the base manifold M:

$$\boldsymbol{g} = g_{\mu\nu}\boldsymbol{d}x^{\mu} \otimes \boldsymbol{d}x^{\nu} \tag{1.183}$$

Defining an orthonormal frame at a point p gives rise to the tetrad formulation of general relativity, sometimes referred to as **vierbiens**.

Taking the soldering form and pulling it back to the base Lorentzian manifold via a local section forms a tetrad basis (obviously defining local coordinates). The soldering form can be expressed in terms of a natural basis of $\mathbb{R}^4$, where $dim(M) = 4$, only looking at the components of the soldering form:

$$s^*\theta^i\left(\boldsymbol{X}\right) \mapsto e^i \in \Lambda^p\left(T_p^*\mathcal{M}, \mathbb{R}^4\right) = e^i{}_{\mu}\left(\boldsymbol{X}\right)dx^{\mu} \tag{1.184}$$

$$\Rightarrow \boldsymbol{g}\left(\boldsymbol{X}, \boldsymbol{Y}\right) = \eta_{ij}s^*\theta^i\left(\boldsymbol{X}\right) \otimes s^*\theta^j\left(\boldsymbol{Y}\right) \tag{1.185}$$

$$\Rightarrow e^i{}_{\mu}e^j{}_{\nu}\eta_{ij} = g_{\mu\nu} \tag{1.186}$$

Similarly one can define a dual vector basis $\boldsymbol{e_i} = e_i{}^{\mu}\boldsymbol{\partial_{\mu}}$ for the tetrads, where $e_a{}^{\mu}e^a{}_{\nu} = \delta^{\mu}_{\nu}$.

Additionally, the connection components pulled back from the frame bundle define the **spin connection** on the Lorentzian manifold. The spin connection is related to the Levi-Civita connection via the soldering form:

$$s^*\boldsymbol{\omega}^i{}_j = (\omega^s)^i_{\mu j}\boldsymbol{d}x^{\mu} \in \Lambda^p\left(T_p^*\mathcal{M}, \mathfrak{so}_{3,1}\right) \tag{1.187}$$

Now following the same procedure as 1.149 a gauge transformation is as follows:

$$(\omega_2^s)^i_{\mu j} = \left(\Lambda^{-1}\right)^i{}_k(\omega_1^s)^k_{\mu l}\left(\Lambda\right)^l{}_j + \left(\Lambda^{-1}\right)^i{}_k\partial_{\mu}\left(\Lambda\right)^k{}_j \tag{1.188}$$

The above is equivalent to saying the tetrads transform under a gauge transformation $\boldsymbol{\Lambda^{-1}e}, \boldsymbol{\Lambda} \in SO(3,1)$, this is the first instance of an Einstien Cartan/gauge theory of gravity. Note, significantly the Lorentz transformations now have a position dependence, revealing a hidden guage redundancy in general relativity.

To prove this, first take torsion expressed on the base Lorentzian manifold:

$$\begin{align}
&de^a + (\omega^s)^a{}_b \wedge e^b = 0 \tag{1.189}\\
\Leftrightarrow &-\partial_\mu e^a{}_\nu dx^\mu \wedge dx^\nu = (\omega^s)^a_{\mu b} e^b{}_\nu dx^\mu \wedge dx^\nu \tag{1.190}\\
&-\left[\nabla_\mu e^a{}_\nu + \cancel{\Gamma^\lambda_{\mu\nu} e^a{}_\lambda}\right] dx^\mu \wedge dx^\nu = (\omega^s)^a_{\mu b} e^b{}_\nu dx^\mu \wedge dx^\nu \tag{1.191}\\
&-\delta^\rho_\nu \nabla_\mu e^a{}_\rho dx^\mu \wedge dx^\nu = (\omega^s)^a_{\mu b} e^b{}_\nu dx^\mu \wedge dx^\nu \tag{1.192}\\
&e^a{}_\rho e^b{}_\nu \nabla_\mu e_b{}^\rho dx^\mu \wedge dx^\nu = (\omega^s)^a_{\mu b} e^b{}_\nu dx^\mu \wedge dx^\nu \tag{1.193}\\
\Rightarrow &e^a{}_\rho \nabla_\mu e_b{}^\rho = (\omega^s)^a_{\mu b} \tag{1.194}\\
\Rightarrow &\Gamma^\rho_{\mu\nu} = e_a{}^\rho \left(\partial_\mu e^a{}_\nu + (\omega^s)^a_{\mu b} e^b{}_\nu\right) \tag{1.195}
\end{align}$$

The anti-symmetry of the wedge product and the fact that the Levi Civita connection is symmetric was used. Pushing this a step further by transforming the tetrads as follows:

$$\begin{align}
(\omega^s_2)^a_{\mu b} &= \bar{e}^a{}_\rho \nabla_\mu \bar{e}_b{}^\rho \tag{1.196}\\
&= \left(\Lambda^{-1}\right)^a{}_c e^c{}_\rho \nabla_\mu \left[(\Lambda)^d{}_b e_d{}^\rho\right] \tag{1.197}\\
&= \left(\Lambda^{-1}\right)^a{}_c e^c{}_\rho \nabla_\mu \left[e_d{}^\rho\right] (\Lambda)^d{}_b + \left(\Lambda^{-1}\right)^a{}_c e^c{}_\rho \nabla_\mu \left[(\Lambda)^d{}_b\right] e_d{}^\rho \tag{1.198}\\
&= \left(\Lambda^{-1}\right)^a{}_c (\omega^s_1)^c_{\mu d} (\Lambda)^d{}_b + \left(\Lambda^{-1}\right)^a{}_c \partial_\mu (\Lambda)^c{}_b \tag{1.199}
\end{align}$$

Hence completing the proof. Note that physics and maths conventions are flipped for a gauge transformation $g \to g^{-1}$. The spin connection also has a corresponding curvature:

$$\boldsymbol{\Omega}^a{}_b = d\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_d \tag{1.200}$$

Which transforms from one local trivialisation (basically a gauge transformation) to another as follows:

$$\boldsymbol{\Omega_j} = Ad_{g_{ij}^{-1}} \circ \boldsymbol{\Omega_i} \tag{1.201}$$

Reformulating the Einstien Hilbert action regarding the spin connection and tetrads, we reach the first-order formulation of gravity, sometimes referred to as the tetradic Palitini action [27]. The action changes as follows.

$$S = \int d^4x\sqrt{-g}R\left(\Gamma\right) \rightarrow \int d^4xe\Omega_s(\omega) \tag{1.202}$$

Where $\Omega = e_a{}^{\mu}e_b{}^{\nu}\Omega^{ab}$ and $e = \sqrt{-g}$ Both reproduce the same results for standard general relativity but are viewed from different perspectives. The first-order formulation, without torsion, provides a clear way of coupling spinors to general relativity. The first-order formulation of gravity does, however, deviate when torsion is included.

This importantly allows for the Dirac action to be defined in curved space-time.

Chapter 2

Coupling Spinors to Gravity

2.1 Spinors

2.1.1 Spin representation

One must first understand representation theory to incorporate Lorentz algebra into the language of spinors. This can be explained succinctly in the language of covering spaces.
Spinors are not vectors or tensors; hence, they do not transform linearly under the Lorentz group. A simple and powerful description is double covering. To prevent the use of stereographic projections, let's look at spinors in the non-relativistic setting and consider $SO(3)$ and its double cover $SU(2)$. The ideas can be extrapolated up a dimension to $SO(3,1)$ and the double covering $Spin(3,1)$.

$SO(3)$ is topologically equivalent to a three-sphere with equivalent antipodal points. This means paths on the three-sphere loop around themselves. In mathematical terms, the fundamental group of $SO(3)$ is $\mathbb{Z}_2$. What exactly does this mean? Consider Dirac's famous belt with axis rotations along the belt. The space of rotations is represented by a sphere of radius π with equivalent antipodal points along the boundary. The axis twists as it's pushed along the belt; the rotation of the axis represents a vector inside the space of rotations. So, for example, consider a belt with only one twist (2π twist); the path the axis takes is represented by:

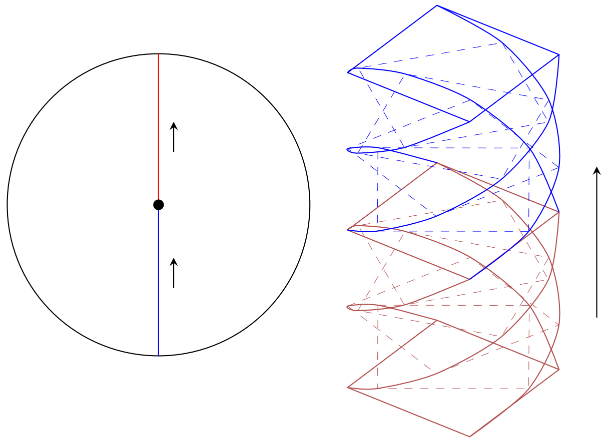

Figure 2.1: The circle (left) is a slice of the rotation phase space as the axis is pushed along the Dirac belt (right)

No matter how much you try to twist the belt, the antipodal points will never touch; hence, in this phase space, a 360-axis rotation along the belt does not equal the identity. I.e. the path in the phase space can never be warped in such a way to allow for a contraction to the trivial path (identity no twisting along the belt). Denoting the 360-axis rotation path as p in the phase space, what happens if you perform a 720-degree rotation (p + p)?

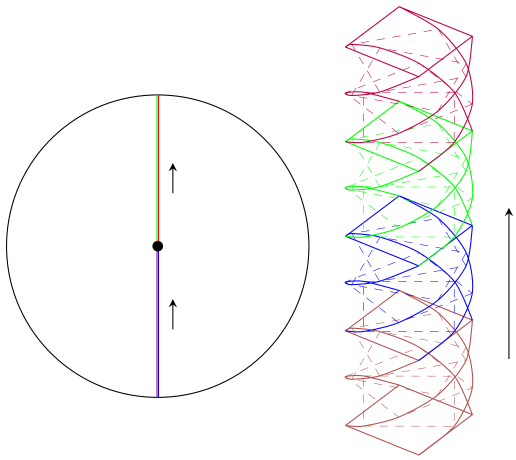

Figure 2.2: The circle (left) is a projective slice of the rotation phase space as the axis is pushed along the Dirac belt (right)

To illustrate, if the paths are deformed, one can see that a 720 rotation along the Dirac belt results in a contractable path. This is because antipodal points now meet at the boundary.

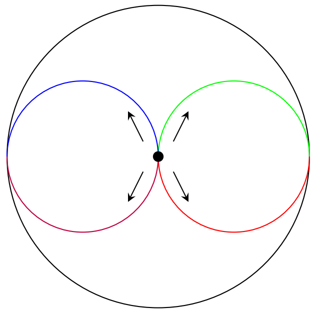

Figure 2.3: The phase space shows a deformed path of the axis, which can be contracted down to the identity

For a 720-axis rotation, antipodal points join at the boundary so that the path can be contracted down to the identity. I.e. a 720 twist in the Dirac belt is equivalent to the belt with no twists. How does this relate to a Spinor? This means the spinor changes state under a $\pi, SO(3)$ transformation. The double cover $SU(2)$ in the fundamental representation is the spin representation of $SO(3)$. There is a 2-1 surjective homomorphism from $SU(2)$ to $SO(3)$. This translates to the Pauli matrices generating the non-relativistic spin representation. Spinors transform correctly under the fundamental representation of $SU(2)$. Similarly, for relativistic spinors, the double covering of $SO(3,1)$, $Spin(3,1)$ is required to transform a spinor properly.

Following this, an additional fibre bundle structure can be used to understand how this spin structure is added to general relativity. A spin bundle is formed from a principle bundle with the structure group Spin(3,1) with a surjective homomorphism to the Orthonormal frame bundle.

2.1.2 Clifford algebra

Consider the anticommutation relation of the Dirac matrices for a general metric:

$$\{\boldsymbol{\gamma_\mu(x)}, \boldsymbol{\gamma_\nu(x)}\} = 2g_{\mu\nu}\mathbb{I} \tag{2.1}$$

Reformulating in the Clifford algebra in terms of Vielbeins [47]:

$$\gamma_\mu(x) = e_\mu{}^a(x)\gamma_a \tag{2.2}$$

$$\Rightarrow \{\boldsymbol{\gamma_a}, \boldsymbol{\gamma_b}\} = 2\eta_{ab}\mathbb{I} \tag{2.3}$$

Where the indices $a, b = 0, 1, 2, 3$. Conveniently, spinors transform in the spinor representation of

$SO(3,1)$. The gamma matrices form the generators of $SO(3,1)$ in the spin representation s.t:

$$S_{ab} = \frac{1}{4}\left[\gamma_a, \gamma_b\right] \tag{2.4}$$

Hence, the Lorentz transformations are characterised by:

$$S[\Lambda(x)] = \exp\left(\frac{1}{2}\lambda^{ab}(x)S_{ab}\right) \tag{2.5}$$

Applying a local Lorentz transformation (all in physics convention) to the Dirac action one must define the covariant derivative [41]:

$$D_\mu\psi = \partial_\mu\psi + \frac{1}{2}\omega_\mu^{ab}S_{ab}\psi \tag{2.6}$$

Leaving the Dirac Lagrangian invariant under a local Lorentz transformation:

$$\psi \to \boldsymbol{S}\left[\boldsymbol{\Lambda}\right]\psi \tag{2.7}$$

$$\boldsymbol{\omega_\mu} \to \boldsymbol{S}[\boldsymbol{\Lambda}]\boldsymbol{\omega_\mu}\boldsymbol{S}[\boldsymbol{\Lambda^{-1}}] + \boldsymbol{S}[\boldsymbol{\Lambda}]\partial_\mu\boldsymbol{S}[\boldsymbol{\Lambda^{-1}}] \tag{2.8}$$

$$D_\mu\psi \to \partial_\mu\left(\boldsymbol{S}[\boldsymbol{\Lambda}]\right)\psi + \boldsymbol{S}[\boldsymbol{\Lambda}]\partial_\mu\psi + \left[\boldsymbol{S}[\boldsymbol{\Lambda}]\boldsymbol{\omega_\mu}\boldsymbol{S}[\boldsymbol{\Lambda^{-1}}] + \boldsymbol{S}[\boldsymbol{\Lambda}]\partial_\mu\boldsymbol{S}[\boldsymbol{\Lambda^{-1}}]\right]\boldsymbol{S}[\boldsymbol{\Lambda}]\psi \tag{2.9}$$

$$\to \boldsymbol{S}[\boldsymbol{\Lambda}]D_\mu \tag{2.10}$$

$$\Rightarrow \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \to \psi^\dagger\boldsymbol{S}[\boldsymbol{\Lambda}]^\dagger\gamma^0\left(i\gamma^\mu\boldsymbol{S}[\boldsymbol{\Lambda}]D_\mu - \boldsymbol{S}[\boldsymbol{\Lambda}]m\right)\psi \tag{2.11}$$

$$\therefore \mathcal{L} \to \mathcal{L} \tag{2.12}$$

Plugging this new spin connection into a generalised Dirac action gives rise to the Dirac action in curved spacetime.

2.1.3 Coupling fermions to gravity

The generalised Dirac action for fermions in curved space-time is [47]

$$S = \int dx^4 e \left(\bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi\right) \tag{2.13}$$

Where $e = \sqrt{-g}$ and the fermions are promoted to Grassman variables. One at this point might be tempted to put this equation on the right-hand side of the Einstien equation; however, fermions/spinors are fundamentally quantum objects; hence, there is no comparison to a macroscopic system. Due to the Pauli exclusion principle, fermions cannot occupy the same energy level with the same spin state; hence, macroscopically sourcing curvature in general relativity with fermions results in a complicated "many-body problem"[41].

This is not to say that curvature cannot be sourced by fermions; for example, neutron stars would have most of the curvature sourced by the fermions, just that the calculations are arduous.

2.2 Utiyama-Sciama-Kibble (ECKS) approach

This approach marries spinors, torsion and general relativity interestingly and intuitively. The approach was developed in three separate papers [44] [35] [20]. Kibble begins his paper by introducing the topic of global Poincare invariance. Consider the Lagrangian transforming under an infinitesimal push:

$$\bar{x}^\mu = x^\mu + \epsilon a^\mu \tag{2.14}$$

$$\bar{\mathcal{L}}(x) = \mathcal{L}(x - \delta x) = \mathcal{L}(x) - \epsilon\partial_\mu\left(a^\mu \mathcal{L}(x)\right) \tag{2.15}$$

$$\delta\mathcal{L} = -\epsilon\partial_\mu\left(a^\mu \mathcal{L}(x)\right) \tag{2.16}$$

$$\tag{2.17}$$

Where a simple Taylor expansion was performed, under an infinitesimal transformation, the Lagrangian varies by a total derivative, hence if one remembers back to the original derivation of the Lagrangian the term gets waked out when varying the action, so this won't change the equations of motion. Since $\mathcal{L}(x)|_{x_0}^{x^1} = 0$, where x_1 and x_0 are extremal points. Now take a look at the action under rotations, reflections and boosts (Lorentz group) $x^\mu \to \Lambda^\mu{}_\nu x^\nu$:

$$\mathcal{L}(x) \to \bar{\mathcal{L}}(\bar{x}) \tag{2.18}$$

$$S = \int \mathcal{L}(x) d^4x = \int \bar{\mathcal{L}}(\bar{x}) d^4\bar{x} \tag{2.19}$$

$$= \int \mathcal{L}(x) \left| \frac{\partial \bar{x}}{\partial x} \right| d^4x \tag{2.20}$$

As you can see the Jacobian evaluates to 1 since det $\left(\frac{\partial \bar{x}}{\partial x}\right) = \pm 1$. Generalising this to the larger $ISO(3,1)$ Poincare group isn't much of a stretch.

Definition (Poincare Group [28]): A d-dimensional group Lie group defining the isometries of a d+1 dimensional Minkowski spacetime. The most straightforward representation of ISO(3,1) comes from a 5x5 matrix representation.

$$ISO(3,1) = \left\{ \left(\begin{array}{c|c} \Lambda & \boldsymbol{a} \\ \hline \mathbf{0} & 1 \end{array} \right) \in GL_5(\mathbb{R}), \;\; \Lambda \in O(3,1) \;\; \boldsymbol{a} \in \mathbb{R}^4 \right\} \tag{2.21}$$

The transformations of the smaller $O(3,1)$ Lorentz group are characterised as follows:

Time reversal: $(t, x, y, z) \to (-t, x, y, z)$

Parity reversal: $(t, x, y, z) \to (t, -x, -y, -z)$

Proper Orthochronous: This is a subgroup, $SO^+(3,1)$, of the Lorentz group characterised by transformations forward in time and restricted by the light cone. Rotations $SO(3)$ and boosts make up this group. Boosts describe the transformation between relative inertial frames.

Poincare algebra fully encapsulates the properties of the group through commutation relations of generators:

$$[P_\mu, P_\nu] = 0 \tag{2.22}$$

$$[M_{\mu\nu}, P_\rho] = \eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu \tag{2.23}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} \tag{2.24}$$

Where M and P represent generators of Lorentz and translation groups, respectively.

Following the first-order formulation found in the context of Supergravity [10], Kibble's work is a local Poincare gauge theory of gravity, where the spin connection transforms in the Lorentz group, and the vierbiens compensate for the space-time translations. In the first-order formalism, the connection $\omega = \omega(e) + K(\psi)$ [10] now includes torsion. First, defining the covariant derivative:

$$\tilde{\nabla}_\mu V_\nu^{ab\cdots c}{}_{de\cdots f} = D_\mu V_\nu^{ab\cdots c}{}_{de\cdots f} - \Gamma^\rho_{\mu\lambda} V_\rho^{ab\cdots c}{}_{de\cdots f} \tag{2.25}$$

$$\Rightarrow \tilde{\nabla}_\mu V_\nu^{ab\cdots c}{}_{de\cdots f} - \tilde{\nabla}_\nu V_\mu^{ab\cdots c}{}_{de\cdots f} = D_\mu V_\nu^{ab\cdots c}{}_{de\cdots f} - D_\nu V_\mu^{ab\cdots c}{}_{de\cdots f} - T_{\mu\nu}{}^\rho V_\rho^{ab\cdots c}{}_{de\cdots f} \tag{2.26}$$

Where V is a mixed Lorentz (p,q) tensor and (1, 0) general vector. D_μ transforms the Lorentz aspect of the tensor and Γ is the usual connection. Explicitly the Lorentz aspect of the mixed tensor transforms as:

$$D_\sigma V_\mu^{ab\cdots c}{}_{de\cdots f} = \partial_\sigma V_\mu^{ab\cdots c}{}_{de\cdots f} \tag{2.27}$$

$$+ \omega^a_{\sigma l} V_\mu^{lb\cdots f}{}_{de\cdots f} + \omega^b_{\sigma l} V_\mu^{al\cdots f}{}_{de\cdots f} \cdots \tag{2.28}$$

$$- \omega^l_{\sigma d} V_\mu^{ab\cdots c}{}_{le\cdots f} - \omega^l_{\sigma e} V_\mu^{ab\cdots c}{}_{dl\cdots f} \cdots \tag{2.29}$$

Also under this new covariant derivative the tetrads, much like the metric, follow a metricity condition, sometimes referred to as the tetrad postulate [10]:

$$\tilde{\nabla}_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu - \Gamma^\sigma_{\mu\nu} e^a{}_\sigma = 0 \tag{2.30}$$

Putting this all together the first formulation of gravity take takes the form:

$$\delta S\left[\boldsymbol{\delta\omega}\right] = \int d^4 x e \delta\Omega \tag{2.31}$$

The variation of the Ω has a neat trick that tidies up the calculation:

$$\delta\boldsymbol{\Omega}^{ab} = \partial_\mu \delta\omega_\nu^{ab} \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu + \delta\left(\omega^a_{\mu c}\omega^{cb}_\nu\right) \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu \tag{2.32}$$

$$= \partial_\mu \delta\omega_\nu^{ab} \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu + \delta\left(\omega^a_{\mu c}\right)\omega^{cb}_\nu + \omega^a_{\mu c}\delta\left(\omega^{cb}_\nu\right) \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu \tag{2.33}$$

$$= D_\mu \delta\omega_\nu^{ab} - \delta\omega_\nu^{ac}\omega^b_{\mu c} - \delta\omega_\nu^{cb}\omega^a_{\mu c} \tag{2.34}$$

$$+ \delta\left(\omega^a_{\mu c}\right)\omega^{cb}_\nu + \omega^a_{\mu c}\delta\left(\omega^{cb}_\nu\right) \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu$$

$$= D_\mu \delta\omega_\nu^{ab} \boldsymbol{dx}^\mu \wedge \boldsymbol{dx}^\nu \tag{2.35}$$

$$\Rightarrow \frac{1}{2}\delta\Omega^{ab}{}_{\mu\nu} = D_{[\mu}\delta\omega^{ab}_{\nu]} \tag{2.36}$$

Note a few techniques have been used here. Firstly the curvature two form is antisymmetric about a, b; hence the terms in 2.35 are cancelled. Additionally, the variation of the spin connection is promoted to a (1,1) Lorentz and (0,1) general mixed tensor for the same reasons as the Levi Civita symbol. Plugging this into the action and using 2.26:

$$\delta S\left[\boldsymbol{\delta\omega}\right] = \int d^4x e e_a{}^\mu e_b{}^\nu 2 D_{[\mu}\delta\omega^{ab}_{\nu]} \tag{2.37}$$

$$= \int d^4x e e_a{}^\mu e_b{}^\nu \left(2\tilde{\nabla}_{[\mu}\delta\omega^{ab}_{\nu]} + T_{\mu\nu}{}^\rho \delta\omega^{ab}_\rho\right) \tag{2.38}$$

$$= \int d^4x e 2\tilde{\nabla}_{[\mu}\left(e_a{}^\mu e_b{}^\nu \delta\omega^{ab}_{\nu]}\right) + e e_a{}^\mu e_b{}^\nu T_{\mu\nu}{}^\rho \delta\omega^{ab}_\rho \tag{2.39}$$

$$= \int d^4x e 2\left(\partial_{[\mu}\left(e_a{}^\mu e_b{}^\nu \delta\omega^{ab}_{\nu]}\right) + \Gamma^\mu_{[\mu\lambda} e_a{}^\lambda e_b{}^\nu \delta\omega^{ab}_{\nu]}\right) + \cdots \tag{2.40}$$

$$= \int d^4x 2\partial_\mu\left(e e_a{}^\mu e_b{}^\nu \delta\omega^{ab}_\nu\right) - \underbrace{2\partial_\mu(e) e_a{}^\mu e_b{}^\nu \delta\omega^{ab}_\nu}_{\partial_\mu e = e\left\{{}^{\lambda}_{\lambda\mu}\right\}} + \cdots \tag{2.41}$$

$$= \int d^4x \left(T^\rho_{\rho a} e_b{}^\nu - T^\rho_{\rho b} e_a{}^\nu + T^\nu_{ab}\right)\delta\omega^{ab}_\nu + 2\partial_\mu\left(e e_a{}^\mu e_b{}^\nu \delta\omega^{ab}_\nu\right) \tag{2.42}$$

At this point, we notice a striking feature. If no spinors couple to gravity, then there is no torsion. Adding spinor content to the Palitini formalism, one finds [10]:

$$\delta S_{\frac{1}{2}}\left[\boldsymbol{\delta\omega}\right] = -\frac{1}{2}\int d^4x e \delta\left(\bar{\Psi}\gamma^\mu \overleftrightarrow{D}_\mu \Psi\right) \tag{2.43}$$

$$= -\frac{1}{2}\int \Psi\gamma^\mu S_{ab}\Psi\delta\omega^{ab}_\mu \tag{2.44}$$

$$\Rightarrow T_{ab}{}^\mu = \frac{1}{2}\kappa^2 \bar{\Psi}\gamma^\mu S_{ab}\Psi \tag{2.45}$$

Note the Lorentzian signature $(- + ++)$ is used here. This induces a sign change in the Dirac action. Plugging this into 1.55 reveals a quartic Ψ term in the action [10]:

$$S = \frac{1}{2} \int \mathrm{d}^4 x e \left[\frac{1}{\kappa^2} R - \bar{\Psi} \gamma^\mu \overleftrightarrow{D}_\mu \Psi + \frac{1}{16} \kappa^2 \left(\bar{\Psi} \gamma_\mu S_{\nu\rho} \Psi \right) \left(\bar{\Psi} \gamma^\mu S^{\nu\rho} \Psi \right) \right] \tag{2.46}$$

Where R is the Riemann curvature tensor constructed from the Levi Civita connection. At this point, one can stop and consider this an $SO(3,1)$ frame bundle theory where the connection is now endowed with torsion. Note this is an equation for the massless Dirac action; the Dirac action with mass takes a more general form. The bundle construction is the same as GR, with the additive of torsion. Professor João Maguejo considered this type of theory in the cosmological setting. The quartic fermion term produces a bounce, preventing the formation of singularity [24] [29].

In his original work, Kibble doesn't stop there and considers the entire Poincare group. Such treatment of the tetrads as gauge fields of spacetime translations is still a hot topic of discussion, with some agreeing that such a formulation is possible and others convinced this is not possible [18].

Following the same arguments Kibble used. Consider the Poincare transformation of the spinor field actively now where spacetime coordinates are fixed. In such a framework, the spinor transforms as:

$$x^\mu \to \bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\nu \tag{2.47}$$

$$\bar{\Psi}(\bar{x}) = \boldsymbol{S}\left[\boldsymbol{\Lambda}\right] \Psi(x) \tag{2.48}$$

For infinitesimal local transformations:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu \tag{2.49}$$

$$\delta x^\mu = \epsilon^\mu{}_\nu x^\nu + \delta a^\mu \tag{2.50}$$

Where $\epsilon^\mu{}_\nu$ is an infinitesimal antisymmetric push. Notice since the infinitesimal transformations are all coordinate dependent $\epsilon(x)x + a(x)$ can be denoted as $\sigma(x)$. Specifically, Hehl in [15] believes this diminishes the role of the Lorentz transformation; however, one must remember these local transformations retain their original properties. $\epsilon(x)$ is still a local Lorentz rotation, and $a(x)$ is still a local translation;

hence there is no issue. Simply Taylor expanding the spinor field about the coordinate reveals [40]:

$$\delta\Psi(x) = \frac{1}{2}\lambda^{ab}S_{ab}\Psi(x) - \delta x^{\nu}\partial_{\nu}\Psi(x) \tag{2.51}$$

$$\partial_{\mu}\delta\Psi = \frac{1}{2}\partial_{\mu}\left(\lambda^{ab}\right)S_{ab}\Psi - \delta x^{\lambda}\partial_{\lambda}\partial_{\mu}\Psi + \frac{1}{2}\lambda^{ab}S_{ab}\partial_{\mu}\left(\Psi\right) - \partial_{\mu}\left(\delta x^{\lambda}\right)\partial_{\lambda}\Psi \tag{2.52}$$

Introducing the same covariant D_{μ} as the other subsection removes the inhomogeneous third term in 2.52 from the last term in 2.8. Finally, Kibble considers a vierbein field to deal with the last term in 2.52. In this sense, the vierbeins act as active general coordinate transformations.

Since the last term is proportional to $\partial_{\mu}\Psi$, one must consider a field that multiplies the covariant derivative. Hence one considers [40]:

$$D_a\Psi = e^{\mu}{}_{a}D_{\mu}\Psi \tag{2.53}$$

$$\text{Where } \delta\left(D_a\Psi\right) = \frac{1}{2}\lambda^{cb}S_{cb}D_a\Psi(x) - \lambda^{b}{}_{a}D_b\Psi - \delta x^{\mu}\partial_{\mu}D_a\Psi \tag{2.54}$$

Now, under this consideration, the Dirac Lagrangian is invariant up to a total derivative $\delta\mathcal{L}_{\Psi} = -\partial_{\mu}\left(\delta x\mathcal{L}_{\Psi}\right)$. Unfortunately, this theory has no direct analogy to the standard view of gauge theories, leading to ambiguities in treating the vierbeins. From the perspective of fibre bundles, the spin connection and tetrad (solder form) cannot transform under different groups; hence, this theory admits a different underlying structure to a gauge theory. Some claim that it is possible to reconcile the translation issue in the context of an affine Poincare bundle [37]; however, this would result in a different mathematical structure proposed by Kibble and Sciama. Instead, the problem can be resolved by considering the symmetry breaking of a larger group.

Chapter 3

Symmetry breaking to gravity

Much like the standard model, one can consider a symmetry-breaking mechanism, where the structure group of a fibre bundle breaks down to $SO(3,1)$ or the more generalised theory including torsion. Following the work of Professor Kellogg Stelle [38][39], Professor Arkady Tseytlin [42], R. F. Sobreiro et al[36], F. W. Hehl [15] and others [19]. The de Sitter groups $SO(p,q)$ where $p+q=5$ requires gauge symmetry breaking so that vierbeins emerge. An extended version of the Higgs mechanism induces symmetry breaking and different actions. In these theories general relativity emerges as a limiting case. The emergence of gravity presents itself through an Inönü-Wigner contraction [17] [23].

Definition (Inönü-Wigner contraction): The mechanism by which a new Lie algebra is obtained by taking the limit of a parameter inside another Lie algebra, which alters the structure constant.

Specifically for this discussion, the de Sitter algebra is spontaneously broken to the Lorentz group $SO(3,1)$. Consider an $SO(3,2)$ frame bundle, with an associated connection ω^{AB}. The generators of the de Sitter group have the following commutation relation [36]:

$$[J^{AB}, J^{CD}] = -\frac{1}{2}\left[\left(\eta^{AC}J^{BD} + \eta^{BD}J^{AC}\right) - \left(\eta^{AD}J^{BC} + \eta^{BC}J^{AD}\right)\right] \tag{3.1}$$

Where $\eta^{AB} = (1,1,1,-1,-1)$. The Yang-Mills curvature follows the same structure as 1.166. The (Anti-)De Sitter group can denoted as $SO(3,2) = SO(3,1) \otimes S(4)$ where $S(4) = SO(3,2)/SO(3,1)$ (cosets) [36]. A convenient way of actualising the decomposition is by projecting out $A=5$ such that $J^{5a} = J^a$

and $a \in \{0,1,2,3\}$. The group commutation relations appear as [36]:

$$\left[J^{ab},J^{cd}\right] = -\frac{1}{2}\left[\left(\eta^{ac}J^{bd}+\eta^{bd}J^{ac}\right)-\left(\eta^{ad}J^{bc}+\eta^{bc}J^{ad}\right)\right] \tag{3.2}$$

$$\left[J^{a},J^{b}\right] = \frac{1}{2}J^{ab}, \tag{3.3}$$

$$\left[J^{ab},J^{c}\right] = \frac{1}{2}\left(\eta^{ac}J^{b}-\eta^{bc}J^{a}\right) \tag{3.4}$$

Where $\eta^{ab} \equiv \mathrm{diag}(1,1,1,-1)$. As you can see this group decomposition has a striking similarity with the Poincare group. The. Additionally, the connection transforms as:

$$\boldsymbol{\omega} \to g^{-1}\left(\frac{1}{\kappa}\boldsymbol{d}+\boldsymbol{\omega}\right)g \tag{3.5}$$

Notice an arbitrary dimensionless parameter next to the exterior derivative. This will become an integral component of the Inönü-Wigner contraction.

Since the connection is lie algebra-valued, it can be decomposed in terms of the generators as follows [36]:

$$\boldsymbol{\omega} = \omega_{AB}\boldsymbol{J}^{AB} \tag{3.6}$$

$$= \omega_{ab}\boldsymbol{J}^{ab} + \theta^{a}\boldsymbol{J}_{a} \tag{3.7}$$

Notice these $\mathfrak{so}(3,2)$ indices, not the matrix indices seen before. Infinitesimally expanding $\boldsymbol{g} = \exp(\kappa\boldsymbol{\zeta}) \approx \mathbb{I}+\kappa\boldsymbol{\zeta}$ and decomposing $\boldsymbol{\zeta} = \alpha_{ab}\boldsymbol{J}^{ab}+\xi_{a}\boldsymbol{J}^{a}$:

$$(\mathbb{I}-\kappa\boldsymbol{\zeta})\left(\frac{1}{\kappa}\boldsymbol{d}+\boldsymbol{\omega}\right)(\mathbb{I}+\kappa\boldsymbol{\zeta}) = \boldsymbol{\omega}+\boldsymbol{d\zeta}+\kappa\boldsymbol{\omega\zeta}-\kappa\boldsymbol{\zeta\omega}-\cancel{\kappa\boldsymbol{\zeta d\zeta}}-\cancel{\kappa^{2}\boldsymbol{\zeta\zeta}} \tag{3.8}$$

$$\Rightarrow \boldsymbol{\omega} \to \boldsymbol{\omega}+\boldsymbol{d\zeta}+\kappa\left[\boldsymbol{\omega},\boldsymbol{\zeta}\right] = \boldsymbol{\omega}+\boldsymbol{\mathcal{D}\zeta} \tag{3.9}$$

$$\Leftrightarrow \boldsymbol{A}+\boldsymbol{\theta}+\boldsymbol{d}\left(\boldsymbol{\alpha}+\boldsymbol{\xi}\right)+\kappa\left[\boldsymbol{A}+\boldsymbol{\theta},\boldsymbol{\alpha}+\boldsymbol{\xi}\right] \tag{3.10}$$

$$\boldsymbol{A}+\boldsymbol{\theta}+\boldsymbol{D}\left(\boldsymbol{\alpha}+\boldsymbol{\xi}\right)+\kappa\left[\boldsymbol{\theta},\boldsymbol{\alpha}+\boldsymbol{\xi}\right] \tag{3.11}$$

Where $\boldsymbol{D}\cdot = \boldsymbol{d}\cdot + \kappa\left[\boldsymbol{A},\cdot\right]$ Plugging the commutation relations for De Sitter group into 3.11 one arrives at [36]:

$$A_{ab} \longmapsto A_{ab} + \mathrm{D}\alpha_{ab} + \frac{\kappa}{4}\left(\theta_a \xi_b - \theta_b \xi_a\right) \tag{3.12}$$

$$\theta^a \longmapsto \theta^a + \mathrm{D}\xi^a + \kappa\alpha^{ab}\theta_b. \tag{3.13}$$

The contraction to the Poincare group manifests itself by including a mass parameter m such that [36]:

$$A \rightarrow \kappa^{-1} A \tag{3.14}$$

$$\theta \rightarrow \kappa^{-1} m\theta \tag{3.15}$$

With this new parameter, the commutation relations of the De-sitter change as follows [36]:

$$\left[J^{ab}, J^{cd}\right] = -\frac{1}{2}\left[\left(\eta^{ac}J^{bd} + \eta^{bd}J^{ac}\right) - \left(\eta^{ac}J^{bc} + \eta^{bc}J^{ad}\right)\right] \tag{3.16}$$

$$\left[J^a, J^b\right] = \frac{m^2}{2\kappa^2} J^{ab} \tag{3.17}$$

$$\left[J^{ab}, J^c\right] = \frac{1}{2}\left(\eta^{ac}J^b - \eta^{bc}J^a\right) \tag{3.18}$$

When the mass parameter is very small and the coupling parameter is high (occurs at low energies), the De-Sitter algebra undergoes a contraction to the Poincare group. This causes the total transformation to reduce to [36]:

$$A_{ab} \rightarrow A_{ab} + D\alpha_{ab} \tag{3.19}$$

$$\theta^a \rightarrow \theta^a - \alpha^a{}_b\theta^b \tag{3.20}$$

How exactly is this actualised physically? In Professor Kellogg's scheme, a non-polynomial action is introduced; this is because the Pontryagin action is a topological term ($F \wedge F$), where $\boldsymbol{F}^{AB} = \boldsymbol{d\omega}^{AB} + \boldsymbol{\omega}^A{}_C \wedge \boldsymbol{\omega}^C{}_B$. Instead, an auxiliary field y^A is placed in the action [38].

$$\mathcal{L} = m\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{AB} F_{\rho\sigma}{}^{CD} \epsilon_{ABCD} - \lambda(y^A y_A + m^{-2}) \tag{3.21}$$

y^A was first implemented in [4] and comes from spontaneous symmetry breaking in supergravity. $\lambda(x)$ is a scalar density Lagrange multiplier, and m has $[L]^{-1}$, where L denotes units of length and $y^A y_A = -m^{-2}$ is a constraint.

Depending on the choice of y, the Lagrangian is either SO(3,1) or Poincare invariant. Choosing $y_0 = (0,0,0,0,m^{-1})$ the effective lagrangian is [40]:

$$\mathcal{L} = \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^{ab} F_{\rho\sigma}{}^{cd} \epsilon_{abcd} \tag{3.22}$$

Where the lowercase Latin indices run from zero to 3. This implies a symmetry breaking from $SO(3,2) \to SO(3,1)$. However, a more general gauge can be chosen. The generator J^{ab} forms the little group $H = SO(3,1)$. An element $g \in SO(3,2)$ near the identity can be written as:

$$g = \exp\left(\zeta^a J_a\right) h \tag{3.23}$$

For $h \in H$ and ζ^a is a parameter in the quotient space $SO(3,2)/SO(3,1)$ [38]. The general representation of y^A is defined as [40]:

$$y^a = m^{-1}\left(\zeta^a/\zeta\right)\sinh(m\zeta), \quad y^5 = m^{-1}\cosh(m\zeta) \tag{3.24}$$

Where $\zeta = \zeta^a \zeta_a$ is the Goldstone field produced from the symmetry breaking pattern $SO(3,2) \to SO(3,1)$. The Goldstone field transforms non-linearly, where $\zeta \to \zeta^{'}$ [40]:

$$\mathrm{g_0} \exp\left(\zeta^{\mathrm{a}} \mathrm{P_a}\right) = \exp\left[\zeta^{'\mathrm{a}}\left(\mathrm{g_0}, \zeta\right) \mathrm{P_a}\right] \mathrm{h_0}\left(\ \mathrm{g_0}, \zeta\right) \tag{3.25}$$

where $g_0 \in SO(3,2)$ and $h_0 \in SO(3,1)$. $\zeta^{'}$ and h_0 are non-linear functions.

Consider the field $\Psi(x)$ that transforms linearly under an irreducible representation ρ of $SO(3,2)$ and a field $\bar{\Psi}(x)$ that contains fields transforming ζ, h non-linearly in $SO(3,1)$ indicies.

The relation between these fields is as follows [38]:

$$\bar{\Psi}(x) = \rho\left[\exp\left(\mathrm{i}\zeta^a(x) P_a\right)\right] \Psi(x) \tag{3.26}$$

Using 3.25 $\bar{\Psi}$ transforms as follows [40]:

$$\bar{\Psi}' = \sigma\left[h_1(\zeta, g)\right]\bar{\Psi} \tag{3.27}$$

How does General Relativity (Einstien Cartan variant) emerge from such a structure?

The $SO(3,2)$ gauge potential, $\boldsymbol{\omega}^{AB}$, can be deconstructed into what is referred to as "non-linear realisations"[38] of $SO(3,2)$; the well-known spin connection, $\bar{\boldsymbol{\omega}}^{ab}$ and vierbein fields.

The vierbien and spin connection takes the following form:

$$\bar{e}_\mu{}^a J_a = m^{-1}\bar{\theta}^a_\mu J_a \tag{3.28}$$

$$\bar{e}^a_\mu J_a \to h\bar{e}_\mu{}^a J_a h^{-1} \tag{3.29}$$

$$\bar{\boldsymbol{\omega}} = \boldsymbol{A} \tag{3.30}$$

$$\frac{1}{2}i\bar{\omega}'^{ab}_\mu J_{ab} = h\left(\frac{1}{2}i\bar{\omega}^{ab}_\mu J_{ab}\right)h^{-1} + h\partial_\mu h^{-1} \tag{3.31}$$

A new Cartan curvature is defined from the bared spin connection and tetrad field [40]:

$$\bar{F}^{ab}_{\mu\nu} = \bar{R}_{\mu\nu}{}^{ab} - m^2\left(\bar{e}_\mu{}^a\bar{e}_\nu{}^b - \bar{e}_\nu{}^a\bar{e}_\mu{}^b\right) \tag{3.32}$$

$$\bar{\boldsymbol{R}} = \boldsymbol{d}\bar{\boldsymbol{\omega}} + \bar{\boldsymbol{\omega}} \wedge \bar{\boldsymbol{\omega}} \tag{3.33}$$

$$\bar{F}_{\mu\nu}{}^{a5} = m\left(\bar{D}_\mu\bar{e}_\nu{}^a - \bar{D}_\nu\bar{e}_\mu{}^a\right) \tag{3.34}$$

$$\bar{D}_\mu\bar{e}_\nu{}^a = a_\mu\bar{e}_\nu{}^a + \bar{\omega}_\mu{}^a{}_b\bar{e}_\nu{}^b. \tag{3.35}$$

Replacing the curvature in the effective lagrangrain with $\bar{F}$ reveals the following effective action [36]:

$$S_{eff} = \frac{1}{8\pi G}\int\left[\frac{1}{2\Lambda^2}\bar{F}^{\mathrm{a}}{}_{\mathrm{b}} \star \bar{F}_{\mathrm{a}}{}^{\mathrm{b}} + \bar{F}^{\mathrm{a5}} \star \bar{F}_{\mathrm{a5}} + \frac{1}{2}\epsilon_{\mathrm{abco}}\bar{F}^{\mathrm{ab}}\bar{e}^{\mathrm{c}}\bar{e}^{\mathrm{o}} + \frac{\Lambda^2}{4}\epsilon_{\mathrm{abc}}\bar{e}^{\mathrm{a}}\bar{e}^{\mathrm{b}}\bar{e}^{\mathrm{c}}\bar{e}^{\jmath}\right] \tag{3.36}$$

$$m^2 = \kappa^2/2\pi G$$

Note the Hodge dual is over $\mathbb{M}^4$ The first and second terms are topologically invariant, the third is the standard Einstien-Hilbert term, and the last is the cosmological constant.

3.1 Discussion

In this framework, Kibble's original translational treatment of the vierbien fields is fully actualised via a symmetry-breaking mechanism. The vierbein gauge transformation in this theory is the desired **passive** transformation; hence, this theory is a gauge theory in the purest sense. The rest of Professor Kellogg and Professor West's work details this theory's underlying fibre bundle structure and holonomy by introducing a development operator. Starting from the De-Sitter group and introducing a non-linear symmetry break mechanism, an effective gravitational theory emerges.

These types of induced gravity theories also have some interesting consequences depending on the choice of vierbein. For example, Professor Arkady's [42] paper details instanton solutions that emerge from a topologically invariant term in the action.

The theory is quite rich; however, it is complicated. The non-linear realisation of the De-Sitter is not trivial. Another question still stands: is it possible to quantize theories with non-compact gauge groups? [42] Furthermore, does it make sense to treat gravity as gauge theory?

Chapter 4

Conclusion

This dissertation starts by discussing the meaning of gauge theories in modern theoretical physics, starting from the inception of gauge theories, where Weyl attempted to extend general relativity, to the most successful application of gauge theory in quantum mechanics. The second section highlights the possibility of extending general relativity by considering an antisymmetric component to the connection; in deriving the field equations for a connection with torsion, an overlooked boundary term was calculated and was found to have the same form as the Gibbons-Hawking-York boundary term. The later part of the section underlines the obstacles facing this boundary term. Furthermore, this section solidifies the importance of including boundary terms by calculating the ADM energy for a $3 + 1$ decomposition of general relativity. Without such boundary terms, calculating scattering amplitudes in QFT for something like the region outside a black hole is impossible. The fibre bundle section introduces gauge theories' underlying and hidden structure through the formal language of fibre bundles. General relativity is reformulated using fibre bundles, and a hidden Lorentz symmetry (redundancy) emerges. This hidden symmetry uncovers a greater spin structure, allowing spinors to be coupled to gravity through the spin connection. The penultimate section goes through a modern formulation of gravity with torsion. It shows the source of torsion is spinors whilst highlighting the issue of an active Poincare transformation and treating the vierbein as a compensatory gauge field. Additionally, the role of torsion in gravity isn't clear as there are experimental limitations for observing phenomena at the Planck scale [10] (rather convenient if you want to write an unfalsifiable theory). The final section reinterprets the compensating veirbein field. Induced gravity theories produce veirbeins through a non-linear symmetry-breaking mechanism, much like the Higgs mechanism.

However, a looming question still stands: does it make sense to view gravity as a gauge theory? What can be said with relative certainty is there is a Lorentz redundancy present in the universe, allowing for spinors to be coupled to gravity. In that sense, this dissertation has fulfilled its goal of coupling spinors to gravity. However, Einstein Cartan theories of gravity are still in their infancy, with lots of leeway for exploration. This dissertation is a preliminary for a further joint investigation with Professor João Maguejo into the dynamics of a black hole in the context of torsion and spinors coupled to gravity. This project will be an extension of [24].

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